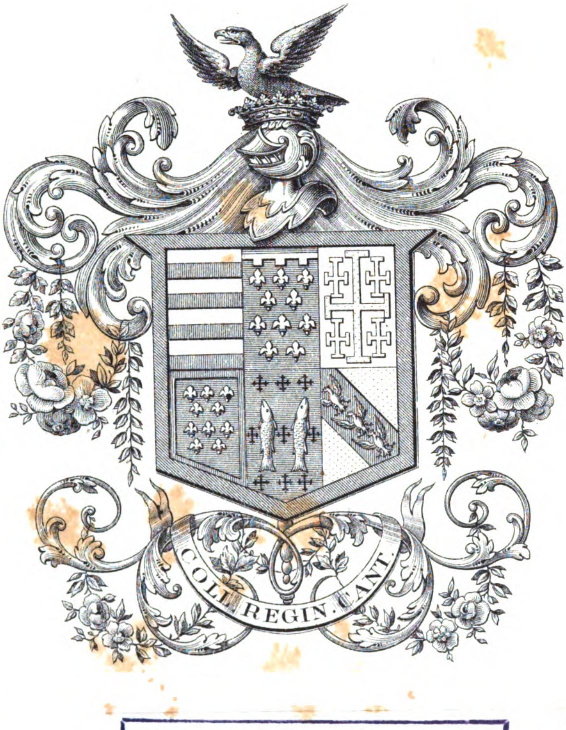
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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XVII.

FROM NOVEMBER, 1885, TO NOVEMBER, 1886.

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LONDON MATHEMATICAL SOCIETY.

VOL. XVII.

TWENTY-SECOND SESSION, 1885–86.

November 12*th*, 1885.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Special Meeting.

The President stated that, in accordance with a notice sent out to members, the meeting had been made a "special" one, for the purpose of considering certain alterations in the "Rules," which were to be proposed by the Council. He then called upon the Treasurer (A. B. Kempe, F.R.S.) to move the adoption of the same. The motion having been seconded by Sir J. Cockle, F.R.S., and carried unanimously, the Meeting then became the

Annual General Meeting.

Mr. L. J. Rogers, B.A., B.Mus., Balliol College, Oxford, was elected a Member.

The Treasurer then read his Report. Its reception was moved by Mr. S. Roberts, seconded by Prof. J. Larmor, and carried unanimously.

At the request of the Chairman, Mr. A. B. Basset consented to act as Auditor.

From the report of the Secretaries, it appeared that the number of members since the last General Meeting, held November 13th, 1884,

had increased from 170 to 181, of these 64 being Life Members. There had been no loss of members by death during the Session.

The following communications had been made:—

On the Theory of Screws in Elliptic Space (Supplementary Notice), and on the Theory of Matrices: A. Buchheim, M.A.

On Sphero-Cyclides: H. M. Jeffery, F.R.S.

Results from a Theory of Transformation of Elliptic Functions: J. Griffiths, M.A.

On the Limits of Multiple Integrals: H. MacColl, B.A.

On the Motion of a Viscous Fluid contained in a Spherical Vessel: Prof. H. Lamb, F.R.S.

On certain Conics connected with a Plane Unicursal Quartic: R. A. Roberts, M.A.

Note on Elliptic Functions, on an Integral Tranformation, and a Theorem in Plane Conics: Asûtosh Mukhopâdhyây, B.A.

On certain Systems of q-Series in Elliptic Functions in which the Exponents in the Numerators and the Denominators are connected by Recurring Relations: the President.

On a Group of Circles connected with the Nine-point Circle: R. Tucker, M.A.

Notes on the Plane Unicursal Quartic: R. A. Roberts, M.A.

The Differential Equations of Cylindrical and Annular Vortices: Prof. M. J. M. Hill, M.A.

On Criticoids: Rev. R. Harley, F.R.S.

Multiplication of Symmetric Functions: Captain P. A. Macmahon, R.A.

Note on Symmetrical Determinants: A. Buchheim, M.A.

Supplementary Paper on Multiple Integrals: H. MacColl, B.A.

On the Binomial Equation $x^p-1=0$: Quinquisection (Second Note): Prof. Cayley, F.R.S.

Sur les figures semblablement variables: Prof. J. Neuberg.

On the extension of Ivory's and Jacobi's Distance-Correspondences for Quadric Surfaces: Prof. J. Larmor, M.A.

Some Properties of a Quadrilateral in a Circle the rectangles under whose opposite sides are equal: R. Tucker, M.A.

On a Method in the Analysis of Plane Curves (Second Paper): J. J. Walker, F.R.S.

On the Geometrical Form of Perfectly Regular Cell-structures: Mrs. Bryant, D.Sc.

On the Constant Quadratic Function of the Inverse Coordinates of $n+1$ Points in Space of n Dimensions: Prof. Sylvester, F.R.S.

On the Flexure of Beams: Prof. K. Pearson, M.A.

Two Elementary Proofs of the Contact of the "N. P." Circle of a Plane Triangle with the In- and Ex-Circles, together with a Property of the Common Tangent: Rev. T. C. Simmons, M.A.

New Relations between Bipartite Functions and Determinants, with a Proof of Cayley's Theorem in Matrices: Dr. T. Muir, M.A.

On Eliminants and Associated Roots: E. B. Elliott, M.A.

On Five Properties of certain solutions of a Differential Equation of the Second Order: Dr. Routh, F.R.S.

On the Arguments of Points on a Surface: R. A. Roberts, M.A.

On Congruences of the Third Order and Class: Dr. Hirst, F.R.S.

An Application of Determinants to the Solution of certain types of Simultaneous Equations: Rev. T. C. Simmons, M.A.

On Binodal Quartics: H. M. Jeffery, F.R.S.

On the Flow of Electricity in a System of Linear Conductors: Prof. J. Larmor, M.A.

On the Potential of an Electrified Spherical Bowl, and on the Velocity Potential due to the Motion of an Infinite Liquid about such a Bowl: A. B. Basset, M.A.

Note on the Porism of the Inscribed and Circumscribing Polygon: L. J. Rogers, B.A.

Liaison Géométrique entre les sphères osculatrices de deux courbes qui ont les mêmes normales principales: Prof. A. Mannheim.

Minor communications were made by the President, the Treasurer, and G. Heppel, M.A.

Some Notes on Quadric Transformations, by the late Mr. Spottiswoode, P.R.S., were edited for the Council by Prof. Cayley, F.R.S.

Additional Exchanges of *Proceedings* were made with the Canadian Institute, Toronto, and the "École Polytechnique de Delft."

The same Journals had been subscribed for as in the preceding Session.

The meeting next proceeded to the election of the new Council. The Scrutators (Mr. G. Heppel and the Rev. T. R. Terry), having examined the Balloting Lists, declared the following gentlemen duly elected:—

President, J. W. L. Glaisher, F.R.S.; Vice-Presidents, Dr. Henrici, F.R.S., Prof. Sylvester, F.R.S., J. J. Walker, F.R.S.; Treasurer, A. B. Kempe, F.R.S.; Secretaries, M. Jenkins, M.A., R. Tucker, M.A. Other Members: Prof. Cayley, F.R.S., Sir J. Cockle, F.R.S., E. B. Elliott, M.A., Prof. Greenhill, M.A., J. Hammond, M.A., Prof. H. Hart, M.A., C. Leudesdorf, M.A., Capt. P. A. Macmahon, R.A., S. Roberts, F.R.S.

Mr. Glaisher thanked the members for his re-election.

The following communications were then made:—

On Waves propagated along the Plane Surface of an Elastic Solid: Lord Rayleigh.

On the Application of Clifford's Graphs to Ordinary Binary Quantics: Mr. Kempe.

On Clifford's Theory of Graphs: Mr. Buchheim.

On Unicursal Curves: Mr. R. A. Roberts.

On some Consequences of the Transformation Formula $y = \sin(L+A+B+C+\dots)$: Mr. J. Griffiths.

The following presents were received:—

"Educational Times," for November.

"Mathematics from the 'Educational Times,'" Vol. XLIII.

"Physical Society—Proceedings," Vol. VII., Pt. II., October, 1885.

"Johns Hopkins University Circulars," Vol. IV., No. 42.

"Extensions of Certain Theorems of Clifford and of Cayley in the Geometry of *n* Dimensions," by E. H. Moore, jun.: (from the *Transactions* of the Connecticut Academy, Vol. VII., 1885).

"Bulletin des Sciences Mathématiques et Astronomiques," T. IX., November, 1885.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. I., Fasc. 21, 22, and 23.

"Acta Mathematica," VII., 2.

"Beiblätter zu den Annalen der Physik und Chemie," B. IX., St. 9 and 10.

"Memorie del R. Istituto Lombardo," Vol. XV., Fasc. 2 and 3.

"R. Istituto Lombardo—Rendiconti," Ser. II., Vols. XVI. and XVII.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VI., No. 3; Coimbra.

"Keglesnitslaeren i Oldtiden," af. H. G. Zeuthen; 4to, Copenhagen, 1885. (Vidensk. Selsk. Skr. 6te Række Naturvidenskabelig og Mathematisk afd. 3de, Bd. I.)

On Waves Propagated along the Plane Surface of an Elastic Solid. By Lord Rayleigh, D.C.L., F.R.S.

[*Read November* 12*th*, 1885.]

It is proposed to investigate the behaviour of waves upon the plane free surface of an infinite homogeneous isotropic elastic solid, their character being such that the disturbance is confined to a superficial region, of thickness comparable with the wave-length. The case is thus analogous to that of deep-water waves, only that the potential energy here depends upon elastic resilience instead of upon gravity.*

Denoting the displacements by α, β, γ, and the dilatation by θ, we have the usual equations

$$\rho \frac{d^2\alpha}{dt^2} = (\lambda+\mu)\frac{d\theta}{dx} + \mu\nabla^2\alpha, \text{ \&c.} \qquad (1),$$

in which

$$\theta = \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} \qquad (2).$$

If α, β, γ all vary as e^{ipt}, equations (1) become

$$(\lambda+\mu)\frac{d\theta}{dx} + \mu\nabla^2\alpha + \rho p^2\alpha = 0, \text{ \&c.} \qquad (3).$$

---

* The statical problem of the deformation of an elastic solid by a harmonic application of pressure to its surface has been treated by Prof. G. Darwin, *Phil. Mag.*, Dec., 1882. [Jan. 1886.—See also *Camb. Math. Trip. Ex.*, Jan. 20, 1875, Question IV.]

Differentiating equations (3) in order with respect to x, y, z, and adding, we get

$$(\nabla^2+h^2)\,\theta = 0 \quad\text{.............................(4)},$$

in which $$h^2 = \rho p^2 / (\lambda+2\mu) \quad\text{.........................(5)}.$$

Again, if we put $$k^2 = \rho p^2 / \mu \quad\text{.............................(6)},$$

equations (3) take the form

$$(\nabla^2+k^2)\,\alpha = \left(1-\frac{k^2}{h^2}\right)\frac{d\theta}{dx},\ \text{\&c.}\quad\text{..................(7)}.$$

A particular solution of (7) is*

$$\alpha = -\frac{1}{h^2}\frac{d\theta}{dx},\quad \beta = -\frac{1}{h^2}\frac{d\theta}{dy},\quad \gamma = -\frac{1}{h^2}\frac{d\theta}{dz}\quad\text{............(8)};$$

in order to complete which it is only necessary to add complementary terms u, v, w satisfying the system of equations

$$(\nabla^2+k^2)\,u = 0,\quad (\nabla^2+k^2)\,v = 0,\quad (\nabla^2+k^2)\,w = 0\quad\text{.........(9)},$$

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = 0\quad\text{.........................(10)}.$$

For the purposes of the present problem we take the free surface as the plane $z = 0$, and assume that, as functions of x and y, the displacements are proportional to e^{ifx}, e^{igy}. Thus (4) takes the form

$$\left(\frac{d^2}{dz^2}+h^2-f^2-g^2\right)\theta = 0;$$

so that $$\theta = Pe^{-rz}+Qe^{+rz}\quad\text{..........................(11)},$$

where $$r^2 = f^2+g^2-h^2\quad\text{.........................(12)}.$$

In (11), r is supposed to be real; otherwise the dilatation would penetrate to an indefinite depth. For the same reason, we must retain only that term (say the first) for which the exponent is negative within the solid.† Thus $Q = 0$, and we will write for brevity $P = 1$, or rather $P = e^{ipt}\,e^{ifx}\,e^{igy}$, but the exponential factors may often be omitted without risk of confusion, so that we may take

$$\theta = e^{-rz}\quad\text{..............................(13)}.$$

* Lamb on the Vibrations of an Elastic Sphere, *Math. Soc. Proc.*, May, 1882.

† By discarding these restrictions we may deduce the complete solution applicable to a plate, bounded by parallel plane free surfaces; but I have not obtained any results which seem worthy of quotation.

At the same time the particular solution becomes

$$\alpha = -\frac{if}{h^2} e^{-rz}, \quad \beta = -\frac{ig}{h^2} e^{-rz}, \quad \gamma = \frac{r}{h^2} e^{-rz} \quad \text{.........(14)}.$$

For the complementary terms, which must also contain e^{ifx}, e^{igy} as factors, equations (9) become

$$\left(\frac{d^2}{dz^2} + k^2 - f^2 - g^2\right) u = 0, \text{ \&c} \quad \text{....................(15)};$$

whence, as before, on the assumption that the disturbance is limited to a superficial stratum,

$$u = Ae^{-sz}, \quad v = Be^{-sz}, \quad w = Ce^{-sz} \quad \text{..................(16)},$$

where

$$s^2 = f^2 + g^2 - k^2 \quad \text{............................(17)}.$$

In order to satisfy (10), the coefficients in (16) must be subject to the relation

$$ifA + igB - sC = 0 \quad \text{.........................(18)}.$$

The complete values of α, β, γ may now be written

$$\alpha = -\frac{if}{h^2} e^{-rz} + Ae^{-sz}, \quad \beta = -\frac{ig}{h^2} e^{-rz} + Be^{-sz}, \quad \gamma = \frac{r}{h^2} e^{-rz} + Ce^{-sz} \quad \text{...(19)},$$

in which A, B, C are subject to (18); and the next step is to express the boundary conditions for the free surface. The two components of tangential stress must vanish, when $z = 0$, and these are proportional to

$$\frac{d\beta}{dz} + \frac{d\gamma}{dy}, \quad \frac{d\gamma}{dx} + \frac{d\alpha}{dz}$$

respectively. Hence

$$sB = \frac{2igr}{h^2} + igC, \quad sA = \frac{2ifr}{h^2} + ifC \quad \text{..............(20)}.$$

Substituting from (20) in (18), we find

$$C(s^2 + f^2 + g^2) h^2 + 2r(f^2 + g^2) = 0 \quad \text{..............(21)}.$$

We have still to introduce the condition that the normal traction is zero at the surface. We have, in general,

$$N_3 = \lambda\theta + 2\mu \frac{d\gamma}{dz};$$

or, if we express λ in terms of μ, h, k,

$$N_3 = \mu\left\{\left(\frac{k^2}{h^2}-2\right)\theta+2\frac{d\gamma}{dz}\right\};$$

so that the condition is

$$k^2-2h^2-2\,(+r^2+h^2sC)=0,$$

or, on substitution for r^2 of its value from (12),

$$k^2-2\,(f^2+g^2)-2h^2sC=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(22).$$

By eliminating C between (21) and (22), we obtain the equation by which the time of vibration is determined as a function of the wave-lengths and of the properties of the solid. It is

$$\{k^2-2\,(f^2+g^2)\}\,\{s^2+f^2+g^2\}+4rs\,(f^2+g^2)=0,$$

or, by (17), $\qquad \{2\,(f^2+g^2)-k^2\}^2=4rs\,(f^2+g^2)\quad\ldots\ldots\ldots\ldots\ldots(23).$

If we square (23), and introduce the values of r^2 and s^2 from (12), (17), we get

$$\{2\,(f^2+g^2)-k^2\}^4=16\,(f^2+g^2)^2\,(f^2+g^2-h^2)(f^2+g^2-k^2).$$

As f and g occur here only in the combination (f^2+g^2), a quantity homogeneous with h^2 and k^2, we may conveniently replace (f^2+g^2) by unity. Thus

$$k^8-8k^6+24k^4-16k^2-16h^2k^2+16h^2=0\ldots\ldots\ldots\ldots(24).$$

Since the ratio $h^2 : k^2$ is known, this equation reduces to a cubic and determines the value of either quantity.

If the solid be incompressible ($\lambda=\infty$), $h^2=0$, and the equation becomes $\qquad k^6-8k^4+24k^2-16=0\quad\ldots\ldots\ldots\ldots\ldots\ldots(25).$

The real root of (25) is found to be ·91275, and the equation may be written $\qquad (k^2-{\cdot}91275)(k^4-7{\cdot}08725k^2+17{\cdot}5311)=0.$

The general theory of vibrations of stable systems forbids us to look for complex values of k^2, as solutions of our problem, though it would at first sight appear possible with them to satisfy the prescribed conditions by taking such roots of (12), (17), as would make the *real* parts of the exponents in e^{-rz}, e^{-sz} negative. But, referring back to (23), which we write in the form

$$(2-k^2)^2=4rs,$$

or, in the present case of incompressibility, by putting $r = 1$,

$$(2-k^2)^2 = 4s,$$

we see that we are not really free to choose the sign of s. In fact, from the complex values of k^2, viz., $3{\cdot}5436 \pm 2{\cdot}2301i$, we find

$$4s = -2{\cdot}7431 \pm 6{\cdot}8846i\,;$$

so that the real part of s is of the opposite sign to r, and therefore e^{-rz}, e^{-sz} do not both diminish without limit as we penetrate further and further into the solid.

Dismissing then the complex values, we have, in the case of incompressibility, the single solution

$$k^2 = \frac{\rho p^2}{\mu} = {\cdot}91275\,(f^2+g^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26).$$

From (19), (20), (21), we get in general

$$h^2\alpha = f\left\{-e^{-rz}+\frac{2rs}{s^2+f^2+g^2}e^{-sz}\right\} \quad\ldots\ldots\ldots\ldots\ldots(27),$$

$$h^2\beta = ig\left\{-e^{-rz}+\frac{2rs}{s^2+f^2+g^2}e^{-sz}\right\} \quad\ldots\ldots\ldots\ldots\ldots(28),$$

$$h^2\gamma = r\left\{e^{-rz}-\frac{2(f^2+g^2)}{s^2+f^2+g^2}e^{-sz}\right\} \quad\ldots\ldots\ldots\ldots\ldots(29).$$

In the case of incompressibility, we have k^2 given by (26), and

$$r^2 = f^2+g^2, \quad s^2 = {\cdot}08725\,(f^2+g^2).$$

Hence

$$\left.\begin{aligned} h^2\alpha &= if\left\{-e^{-rz}+{\cdot}5433e^{-sz}\right\}e^{ipt}e^{ifx}e^{igy} \\ h^2\beta &= ig\left\{-e^{-rz}+{\cdot}5433e^{-sz}\right\}e^{ipt}e^{ifx}e^{igy} \\ h^2\gamma &= \sqrt{(f^2+g^2)}\left\{e^{-rz}-1{\cdot}840e^{-sz}\right\}e^{ipt}e^{ifx}e^{igy} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(30).$$

If we suppose the motion to be in two dimensions only, we may put $g = 0$; so that $\beta = 0$, and

$$\left.\begin{aligned} h^2\alpha/f &= i\left\{-e^{-fz}+{\cdot}5433e^{-sz}\right\}e^{ipt}e^{ifx} \\ h^2\gamma/f &= \left\{e^{-fz}-1{\cdot}840e^{-sz}\right\}e^{ipt}e^{ifx} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(31),$$

in which

$$k = {\cdot}9554f, \quad s = {\cdot}2954f \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

For a progressive wave we may take simply the real parts of (31). Thus

$$\left.\begin{aligned} h^2\alpha/f &= (e^{-fz}-{\cdot}5433e^{-sz})\sin(pt+fx) \\ h^2\gamma/f &= (e^{-fz}-1{\cdot}840e^{-sz})\cos(pt+fx) \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(33).$$

The velocity of propagation is p/f, or $\cdot 9554\sqrt{(\mu/\rho)}$, in which $\sqrt{(\mu/\rho)}$ is the velocity of purely transverse plane waves. The surface waves now under consideration move, therefore, rather more slowly than these.

From (32), (33), we see that α vanishes for all values of x and t when $e^{(s-f)z} = \cdot 5433$, *i.e.*, when $fz = \cdot 8659$. Thus, if λ' be the wave-length $(2\pi/f)$, the horizontal motion vanishes at a depth equal to $\cdot 1378\lambda'$. On the other hand, there is no finite depth at which the vertical motion vanishes.

To find the motion at the surface itself, we have only to put $z = 0$ in (33). We may drop at the same time the constant multiplier (h^2/f) which has no present significance. Accordingly,

$$\left.\begin{aligned} \alpha &= \cdot 4567 \sin(pt+fx) \\ \gamma &= -\cdot 840 \cos(pt+fx) \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34),$$

showing that the motion takes place in elliptic orbits, whose vertical axis is nearly the double of the horizontal axis.

The expressions for stationary vibrations may be obtained from (30) by addition to the similar equations obtained by changing the sign of p, and similar operations with respect to f and g. Dropping an arbitrary multiplier, we may write

$$\left.\begin{aligned} \alpha &= -f\left\{-e^{-rz}+\cdot 5433e^{-sz}\right\}\cos pt \sin fx \cos gy \\ \beta &= -g\left\{-e^{-rz}+\cdot 5433e^{-sz}\right\}\cos pt \cos fx \sin gy \\ \gamma &= r\left\{e^{-rz}-1\cdot 840e^{-sz}\right\}\cos pt \cos fx \cos gy \end{aligned}\right\} \quad\ldots\ldots\ldots(35),$$

in which $\qquad r = \sqrt{(f^2+g^2)}, \quad s = \cdot 2954\sqrt{(f^2+g^2)} \quad\ldots\ldots\ldots\ldots\ldots(36).$

As before, the horizontal motion vanishes at a depth such that

$$\sqrt{(f^2+g^2)}\, z = \cdot 8659.$$

We will now examine how far the numerical results are affected when we take into account the finite compressibility of all natural bodies. The ratio of the elastic constants is often stated by means of the number expressing the ratio of lateral contraction to longitudinal extension when a bar of the material is strained by forces applied to its ends. According to a theory now generally discarded, this ratio (σ) would be $\frac{1}{4}$; a number which, however, is not far from the truth for a variety of materials, including the principal metals. In the extreme case of incompressibility σ is $\frac{1}{2}$, and there seems to be no theoretical reason why σ should not have any value between this and -1.*

---

* Prof. Lamb, in his able paper, seems to regard all negative values of σ as exclu-

The accompanying table will give an idea of the progress of the values of $k^2/(f^2+g^2)$ as dependent upon λ/μ, or upon σ. It will be observed that the value diminishes continuously with λ, in accordance with a general principle.*

λ	σ	h^2/k^2	$k^2/(f^2+g^2)$	$k/\sqrt{(f^2+g^2)}$
∞	$\frac{1}{2}$	0	·9127	·9554
μ	$\frac{1}{4}$	$\frac{1}{3}$	·8453	·9194
0	0	$\frac{1}{2}$	·7640	·8741
$-\frac{2}{3}\mu$	-1	$\frac{3}{4}$	·4746	·6896

As an example of finite compressibility, we will consider further the second case of the table. From (12), (17),

$$r^2 = \cdot 7182\,(f^2+g^2), \quad r = \cdot 8475\sqrt{(f^2+g^2)},$$

$$s^2 = \cdot 1547\,(f^2+g^2), \quad s = \cdot 3933\sqrt{(f^2+g^2)}.$$

Hence, from (27), (28), (29), in correspondence with (30), we have

$$\left.\begin{aligned} h^2\alpha &= if\left\{-e^{-rz}+\cdot 5773e^{-sz}\right\} e^{ipt}\,e^{ifx}\,e^{igy} \\ h^2\beta &= ig\left\{-e^{-rz}+\cdot 5773e^{-sz}\right\} e^{ipt}\,e^{ifx}\,e^{igy} \\ h^2\gamma &= \cdot 8475\sqrt{(f^2+g^2)}\left\{e^{-rz}-1\cdot 7320e^{-sz}\right\} e^{ipt}\,e^{ifx}\,e^{igy} \end{aligned}\right\} \quad \ldots\ldots(37).$$

For a progressive wave in two dimensions, we shall have

$$\left.\begin{aligned} h^2\alpha/f &= (e^{-rz}-\cdot 5773e^{-sz})\sin(pt+fx) \\ h^2\gamma/f &= (\cdot 8475e^{-rz}-1\cdot 4679e^{-sz})\cos(pt+fx) \end{aligned}\right\} \quad \ldots\ldots\ldots(38).$$

At the surface,

$$\left.\begin{aligned} h^2\alpha/f &= \quad \cdot 4227\sin(pt+fx) \\ h^2\gamma/f &= -\ \cdot 6204\cos(pt+fx) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(39),$$

so that the vertical axes of the elliptic orbits are about half as great again as the horizontal axes.

---

ded *a priori*. But the necessary and sufficient conditions of stability are merely that the resistance to compression ($\lambda+\frac{2}{3}\mu$) and the resistance to shearing (μ) should be positive. In the second extreme case of a medium which resists shear, but does not resist compression, $\lambda = -\frac{2}{3}\mu$, and $\sigma = -1$. The velocity of a dilatational wave is then $\frac{2}{3}$ of that of a distortional plane wave. (Green, *Camb. Trans.*, 1838.) The general value of σ is $\lambda/(2\lambda+2\mu)$.

* *Math. Soc. Proc.*, June, 1873, Vol. IV., p. 359. "Theory of Sound," t. 1, p. 85. Lamb, *loc. cit.*, p. 202.

It is proper to remark that the vibrations here considered are covered by the general theory of spherical vibrations given by Lamb in the paper referred to. But it would probably be as difficult, if not more difficult, to deduce the conclusions of the present paper from the analytical expressions of the general theory, as to obtain them independently. It is not improbable that the surface waves here investigated play an important part in earthquakes, and in the collision of elastic solids. Diverging in two dimensions only, they must acquire at a great distance from the source a continually increasing preponderance.

On some Consequences of the Transformation Formula $y = \sin(L+A+B+C+\ldots)$.

By JOHN GRIFFITHS, M.A.

[*Read Nov.* 12*th*, 1885.]

CONTENTS OF PAPER.

Notation.

In order to avoid repetitions, it is convenient, for the purposes of

this Note, to write

$$\cos A = \frac{1-(1+\alpha'^2)\,x^2}{1-\alpha^2x^2}, \qquad \sin A = \frac{2\alpha' x\sqrt{1-x^2}}{1-\alpha^2x^2},$$

$$\cos B = \frac{1-(1+\beta'^2)\,x^2}{1-\beta^2x^2}, \text{ \&c.},$$

where $$\alpha^2+\alpha'^2 = 1 = \beta^2+\beta'^2 = \dots.$$

$$\sin L = \frac{(1+k')\,x\sqrt{1-x^2}}{\sqrt{1-k^2x^2}}, \quad \cos L = \frac{1-(1+k')\,x^2}{\sqrt{1-k^2x^2}}.$$

$$u_0 = \frac{sK}{n}, \quad v_0 = \frac{2t-1}{2n}K, \qquad (\text{mod. } k),$$

$$u_0' = \frac{sK'}{n}, \quad v_0' = \frac{2t-1}{2n}K', \qquad (\text{mod } k');$$

s being an integer from 1 to $n-1$, and t an integer from 1 to n.

$(1, x^2)^n \equiv$ rational and integral function of x of the order $2n$. (Prof. Cayley's notation).

$\Pi \equiv$ product of a certain number of factors; for example,

$$\Pi\left\{1-\frac{x^2}{\text{sn}^2 u_0}\right\} = \left(1-\frac{x^2}{\text{sn}^2\frac{K}{n}}\right)\left(1-\frac{x^2}{\text{sn}^2\frac{2K}{n}}\right)\dots\dots\left(1-\frac{x^2}{\text{sn}^2\frac{n-1}{n}K}\right)$$

to $n-1$ factors.

In the case of $n=1$, Π must be taken $\equiv 1$. $\text{ct} \equiv \text{cn} \div \text{sn}$.

The object of the Note is, in the first instance, to show how the three even rational transformation equations of the forms

$$y = \frac{x\,(1, x^2)^{n-1}}{(1, x^2)^n}, \quad y = \frac{(1, x^2)^n}{(1, x^2)^n}, \quad y = \frac{(1, x^2)^n}{x\,(1, x^2)^{n-1}},$$

can be derived from the formula

$$y = \sin(L+A+B+\dots);$$

and, secondly, to notice some results as regards the transformation and complete multiplication by $2n$ of the sn and Θ functions.

SECT. 1.—*On some Developments of the Equation*
$y = \sin(L+A+B+\ldots)$.

Here $y = \dfrac{x\sqrt{1-x^2}}{\sqrt{1-k^2x^2}} \times$ rational function of x, and $\sqrt{1-y^2} =$ rational function of $x \div \sqrt{1-k^2x^2}$, so that Jacobi's change of y, x into $\dfrac{1}{\lambda y}, \dfrac{1}{kx}$ is applicable. In fact, the integral equation gives rise to the differential relation

$$\frac{dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}} = M\,\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}},$$

provided that

$$M\sqrt{1-\lambda^2y^2} = \sqrt{1-k^2x^2}\left\{1+\frac{k'}{1-k^2x^2}+2\Sigma\frac{a'}{1-a^2x^2}\right\}.$$

(See a note by the present writer, *Proc. Lond. Math. Soc.*, Vol. xv., p. 64.)

But, besides the change of y, x in question, we have, in this instance, the following ones, viz., of

$$\left.\begin{array}{l} y \text{ into } \lambda y \\ \lambda \text{ into } \dfrac{1}{\lambda} \\ M \text{ into } M\lambda \end{array}\right\}.$$

Writing $\quad y = \operatorname{sn}(Mu, \lambda) \quad$ and $\quad x = \operatorname{sn}(u, k)$,

we deduce then the groups of formulae,

$$y = Mx\sqrt{1-x^2}\,\Pi\left\{1-\frac{x^2}{\operatorname{sn}^2 u_0}\right\} \div$$

$$\sqrt{1-y^2} = \Pi\left\{1-\frac{x^2}{\operatorname{sn}^2 v_0}\right\} \div$$

$$\sqrt{1-\lambda^2y^2} = \Pi\left\{1-k^2\operatorname{sn}^2 v_0\, x^2\right\} \div$$

$$\text{common denom. } \sqrt{1-k^2x^2}\,\Pi\left\{1-k^2\operatorname{sn}^2 u_0\, x^2\right\}$$

(see notation).

$$\frac{M\lambda}{k^2}\,.\operatorname{sn}(Mu, \lambda) = \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}\left\{1+2\Sigma\frac{(-)^s\operatorname{cn}^2\dfrac{sK}{n}}{1-k^2\operatorname{sn}^2\dfrac{sK}{n}\operatorname{sn}^2 u}\right\},$$

$$M\lambda\,.\,\mathrm{cn}\,(Mu,\,\lambda) = \mathrm{dn}\,u\left\{1+\frac{(-)^n k'}{\mathrm{dn}^2 u}+2\,\Sigma\,\frac{(-)^s\,\mathrm{dn}\,\dfrac{sK}{n}}{1-k^2\,\mathrm{sn}^2\,\dfrac{sK}{n}\,\mathrm{sn}^2 u}\right\},$$

$$M\,.\,\mathrm{dn}\,(Mu,\,\lambda) = \mathrm{dn}\,u\left\{1+\frac{k'}{\mathrm{dn}^2 u}+2\,\Sigma\,\frac{\mathrm{dn}\,\dfrac{sK}{n}}{1-k^2\,\mathrm{sn}^2\,\dfrac{sK}{n}\,\mathrm{sn}^2 u}\right\};$$

where the order of the transformation $= 2n$.

$$M = \Pi\,\mathrm{sn}^2\,\frac{sK}{n} \div \Pi\,\mathrm{sn}^2\,\frac{2t-1}{2n}\,K.$$

SECT. 2.—*Deduction of* $Y = \dfrac{(1,\,x^2)^n}{(1,\,x^2)^n}$ *from the above equations.*

If we put $Y = \sqrt{\dfrac{1-y^2}{1-\lambda^2 y^2}}$, we have, from the foregoing Section, a rational transformation equation, of order $2n$, wherein

$$Y = \Pi\left\{1-\frac{x^2}{\mathrm{sn}^2 v_0}\right\} \div$$

$$\sqrt{1-Y^2} = M\lambda'\,.\,x\sqrt{1-x^2}\;\Pi\left\{1-\frac{x^2}{\mathrm{sn}^2 u_0}\right\} \div$$

$$\sqrt{1-\lambda^2 Y^2} = \lambda'\sqrt{1-k^2x^2}\;\Pi\left\{1-k^2\,\mathrm{sn}^2 u_0\,x^2\right\} \div$$

$$\text{common denom.} = \Pi\left\{1-k^2\,\mathrm{sn}^2 v_0\,x^2\right\}.$$

Here we have, in fact, a formula

$$Y = \cos\,(A+B+C+\ldots) = \frac{(1,\,x^2)^n}{(1,\,x^2)^n},$$

where the coefficients α, β, &c. may be taken to be

$$\alpha = k\,\mathrm{sn}\,\frac{K}{2n},\quad \alpha' = \mathrm{dn}\,\frac{K}{2n},\quad \beta = k\,\mathrm{sn}\,\frac{3K}{2n},$$

and so on for odd multiples of $\dfrac{K}{2n}$.

This gives the differential equation

$$\frac{dY}{\sqrt{1-Y^2\,.\,1-\lambda^2Y^2}} = -\,M\,\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}}.$$

Hence $$\Lambda - \int_0^Y \frac{dY}{\sqrt{1-Y^2 \,.\, 1-\lambda^2 Y^2}} = Mu, \quad \text{or, say,} \quad \Lambda - v_\lambda = Mu.$$

If, then, $\int_0^y \frac{dy}{\sqrt{1-y^2 \,.\, 1-\lambda^2 y^2}} = u_\lambda$ and $\int_0^Y \frac{dY}{\sqrt{1-Y^2 \,.\, 1-\lambda^2 Y^2}} = v_\lambda$,

the relation between the two transformations is

$$u_\lambda + v_\lambda = \Lambda, \qquad (\text{mod } \lambda).$$

SECT. 3.—*Deduction of* $z = \dfrac{x(1, x^2)^{n-1}}{(1, x^2)^n}$.

From $y = \sin(L+A+\dots)$, I have already derived the secondary transformation $\quad iz = \tan(X_0 + X_1 + \dots + X_{n-1})$,

where $\quad \tan X_0 = i\dfrac{(1+k)x}{1+kx^2}, \quad \tan X_1 = \dfrac{2ia_1 x}{1+a_1^2 x^2}$, &c.,

and the coefficients are of the form

$$a_s = \operatorname{dn}\left(\frac{sK'}{n},\ k'\right).$$

See *Proc. Lond. Math. Soc.*, Vol. XVI., p. 90.

This gives $$\int_0^z \frac{dz}{\sqrt{1-z^2 \,.\, 1-\gamma^2 z^2}} = N \int_0^x \frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2 x^2}},$$

or, say, $$u_\gamma = Nu.$$

We have, accordingly,

$$N = \Pi \operatorname{sn}^2\left(\frac{sK'}{n},\ k'\right) \div \Pi \operatorname{sn}^2\left(\frac{2t-1}{2n}K',\ k'\right);$$

$$\operatorname{sn}(Nu, \gamma) = N \operatorname{sn} u \,.\, \Pi\left\{1 + ct^2 u_0' \operatorname{sn}^2 u\right\} \div$$

$$\operatorname{cn}(Nu, \gamma) = \operatorname{cn} u \operatorname{dn} u \,.\, \Pi\left\{1 - \operatorname{dn}^2 u_0' \operatorname{sn}^2 u\right\} \div$$

$$\operatorname{dn}(Nu, \gamma) = \Pi\left\{1 - \operatorname{dn}^2 v_0' \operatorname{sn}^2 u\right\} \div \text{common denom. } \Pi\left\{1 + \operatorname{ct}^2 v_0' \operatorname{sn}^2 u\right\}.$$

(For the values of u_0' and v_0', see notation.)

For example, when $2n = 6$,

$$\operatorname{sn}(Nu, \gamma) = N \operatorname{sn} u \left(1 - k^2 \operatorname{sn}^2 \frac{iK'}{3} \operatorname{sn}^2 u\right)\left(1 - k^2 \operatorname{sn}^2 \frac{2iK'}{3} \operatorname{sn}^2 u\right)$$

$$\div (1+kx^2)\left(1 - k^2 \operatorname{sn}^2 \frac{iK'}{6} \operatorname{sn}^2 u\right)\left(1 - k^2 \operatorname{sn}^2 \frac{5iK'}{6} \operatorname{sn}^2 u\right).$$

SECT. 4.—*Deduction of* $Z = \dfrac{(1, x^2)^n}{x\,(1, x^2)^{n-1}}$.

From Sect. 2 there follows a secondary equation,

$$Z = i\cot(X_1 + X_2 + \dots + X_n), \text{ where } \tan X_1 = \frac{2ib_1x}{1+b_1^2x^2},$$

$$\tan X_2 = \frac{2ib_2x}{1+b_2^2x^2}, \text{ \&c., and } b_t = \text{dn}\left(\frac{2t-1}{2n}K',\ k'\right);$$

t being an integer from 1 to n.

Here $$\int_\infty^Z \frac{dZ}{\sqrt{1-Z^2\,.\,1-\gamma^2Z^2}} = N\int_0^x \frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}},$$

i.e., $$v_\gamma = i\Gamma' + Nu,$$

or $$u_\gamma + v_\gamma = i\Gamma', \quad (\text{mod } \gamma). \qquad \text{See Sect. 3.}$$

SECT. 5.—*Complete Multiplication by* $2n$. *Expression for* sn $2nu$ *deduced from two conjugate Transformations.*

Complete multiplication by $2n$ may be effected in several ways; for instance, writing the equations of Sects. 2 and 4, as $Y = f(x, k)$, $Z = \phi(x, k)$, and taking the modular relations to be

$$\left.\begin{aligned} 2n\Lambda &= MK \\ \Lambda' &= MK' \end{aligned}\right\} \qquad \left.\begin{aligned} \Gamma &= NK \\ 2n\Gamma' &= NK' \end{aligned}\right\},$$

(see *Proc. Lond. Math. Soc.*, Vol. XVI., p. 91,)

we derive $$\int_1^Y \frac{dY}{\sqrt{1-Y^2\,.\,1-\lambda^2Y^2}} = -M\int_0^x \frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}};$$

change Y into X; x into Z; λ, k, M into k, γ, $\dfrac{2n}{N}$; then

$$\int_1^X \frac{dX}{\sqrt{1-X^2\,.\,1-k^2X^2}} = -\frac{2n}{N}\int_0^Z \frac{dZ}{\sqrt{1-Z^2\,.\,1-\gamma^2Z^2}} = -\frac{2n}{N}(i\Gamma' + Nu),$$

i.e., $$\int_0^X \frac{dX}{\sqrt{1-X^2\,.\,1-k^2X^2}} = K - \frac{2n}{N}i\Gamma' - 2nu = K - iK' - 2nu,$$

or $$\text{sn}\,(K - iK' - 2nu) = f(Z, \gamma),$$

where $$Z = \phi(\text{sn}\,u, k).$$

SECT. 6.—*Imaginary Transformations.*

The general form of the transformations of Sect. 3 appears to be

$$\operatorname{sn}(Nu, \gamma) = N \operatorname{sn} u \,.\, \Pi \left\{ 1 - k^2 \operatorname{sn}^2 \frac{2pK + p'iK'}{n} \operatorname{sn}^2 u \right\}$$

$$\div\, \Pi \left\{ 1 - k^2 \operatorname{sn}^2 \frac{2mK + (2m'+1)\, iK'}{2n} \operatorname{sn}^2 u \right\},$$

if p, p', m, m' be certain integers, positive and negative, or, say,

$$\operatorname{sn}(Nu, \gamma) = N \operatorname{sn} u\, (1, \operatorname{sn}^2 u)^{n-1} \div (1, \operatorname{sn}^2 u)^n.$$

The functions $(1, \operatorname{sn}^2 u)^{n-1}$ and $(1, \operatorname{sn}^2 u)^n$ are self-inverse, *i.e.*, each of them remains unaltered, to a factor *près*, when $\dfrac{1}{k \operatorname{sn} u}$ is written therein for $\operatorname{sn} u$.

Since it is essential that

$$\operatorname{cn}(Nu, \gamma) \text{ shall be } = \operatorname{cn} u \operatorname{dn} u \times \text{rational function of } \operatorname{sn} u,$$

the integers must in all cases satisfy the relation

$$\Pi \operatorname{dn}^4 \frac{2miK' - (2m'+1)\, K}{2n} = (k')^2\, \Pi \operatorname{dn}^4 \frac{2piK' - p'K}{n}.$$

There is no difficulty as regards the simple case of $2n = 4$.

Here

$$\operatorname{sn}(Nu, \gamma) = N \operatorname{sn} u \left(1 - k^2 \operatorname{sn}^2 \frac{iK'}{2} \operatorname{sn}^2 u\right)$$

$$\div \left(1 - k^2 \operatorname{sn}^2 \frac{iK'}{4} \operatorname{sn}^2 u\right)\left(1 - k^2 \operatorname{sn}^2 \frac{3iK'}{4} \operatorname{sn}^2 u\right),$$

$$\operatorname{sn}(N_2 u, \gamma_2) = N_2 \operatorname{sn} u \left(1 - k^2 \operatorname{sn}^2 \frac{2K - iK'}{2} \operatorname{sn}^2 u\right)$$

$$\div \left(1 - k^2 \operatorname{sn}^2 \frac{2K - iK'}{4} \operatorname{sn}^2 u\right)\left(1 - k^2 \operatorname{sn}^2 \frac{2K + 3iK'}{4} \operatorname{sn}^2 u\right);$$

$$\operatorname{sn}(N_3 u, \gamma_3) = N_3 \operatorname{sn} u \left(1 - k^2 \operatorname{sn}^2 \frac{4K - iK'}{2} \operatorname{sn}^2 u\right)$$

$$\div \left(1 - k^2 \operatorname{sn}^2 \frac{4K - iK'}{4} \operatorname{sn}^2 u\right)\left(1 - k^2 \operatorname{sn}^2 \frac{4K + 3iK'}{4} \operatorname{sn}^2 u\right),$$

$$\operatorname{sn}(N_4u,\gamma_4) = N_4 \operatorname{sn} u \left(1-k^2 \operatorname{sn}^2 \frac{6K-iK'}{2} \operatorname{sn}^2 u\right)$$

$$\div \left(1-k^2 \operatorname{sn}^2 \frac{6K-iK'}{4} \operatorname{sn}^2 u\right)\left(1-k^2 \operatorname{sn}^2 \frac{6K+3iK'}{4} \operatorname{sn}^2 u\right).$$

Hence

$$\operatorname{am}(4u, k) = \operatorname{am}(Nu,\gamma)+\operatorname{am}(N_2u,\gamma_2)+\operatorname{am}(N_3u,\gamma_3)+\operatorname{am}(N_4u,\gamma_4),$$

$$\operatorname{pm}(4u, k) = \operatorname{am}(Nu,\gamma)-\operatorname{am}(N_2u,\gamma_2)+\operatorname{am}(N_3u,\gamma_3)-\operatorname{am}(N_4u,\gamma_4),$$

if $\qquad \operatorname{sm}(u,k) = k \operatorname{sn}(u,k), \quad \operatorname{am} \equiv \sin^{-1}\operatorname{sn},$

and $\qquad \operatorname{sn}(Nu,\gamma) = N \operatorname{sn} u \left(1-k^2 \operatorname{sn}^2 \frac{iK'}{2} \operatorname{sn}^2 u\right)$

$$\div \left(1-k^2 \operatorname{sn}^2 \frac{iK'}{4} \operatorname{sn}^2 u\right)\left(1-k^2 \operatorname{sn}^2 \frac{3iK'}{4} \operatorname{sn}^2 u\right)$$

$$= (1+\sqrt{k})^2 \frac{\operatorname{sn} u\,(1+k \operatorname{sn} u)}{1+2\sqrt{k}\,(1+\sqrt{k}+k)\operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u};$$

with similar expressions for

$$\operatorname{sn}(N_2u,\gamma_2), \quad \operatorname{sn}(N_3u,\gamma_3), \quad \text{and} \quad \operatorname{sn}(N_4u,\gamma_4),$$

obtained by changing $\sqrt{k}$ into $-i\sqrt{k}$, $-\sqrt{k}$, and $i\sqrt{k}$, respectively.

The multiplication formula in question seems to be true when the order of the transformations $2n$ is of the form 2^m; *i.e.*, in that case we have

$$\operatorname{am}(2nu, k) = \operatorname{am}(Nu,\gamma)+\operatorname{am}(N_2u,\gamma_2)+\operatorname{am}(N_3u,\gamma_3)+\ldots$$
$$\ldots+\operatorname{am}(N_{2n}u,\gamma_{2n}).$$

Also, by the principle of duality, each of the above transformations gives rise to a conjugate one. With regard to these* non-real transformations, which I have considered more fully in an Appendix, it should be mentioned that if λ and γ be the respective moduli corresponding to a pair, then it is necessary to assume relations of the forms

$$\left.\begin{aligned} MK &= 2a\Lambda+2ib\Lambda' \\ MK' &= a'\Lambda'+2ib'\Lambda \end{aligned}\right\} \qquad \left.\begin{aligned} NK &= a'\Gamma+2ib'\Gamma' \\ NK' &= 2a\Gamma'+2ib\Gamma \end{aligned}\right\},$$

where a' is an odd integer, and the other integers are connected by

* This phrase is meant to include both imaginary moduli and those which are real but >1.

the relations $aa'-2b^2 = \pm n$, $b+b' = 0$, if $2n$ = order of the transformations.

[Compare this problem with Jacobi's. See *Fundamenta Nova*, p. 75, and a note thereon by the present writer, *Proc. Lond. Math. Soc.*, Vol. XVI., p. 104.]

SECT. 7.—*Transformation of* Θ *functions. Expressions for* $\Theta(Mu, \lambda)$ *and* $\Theta(Nu, \gamma)$.

The formula for $\Theta(Mu, \lambda)$, corresponding to a first real root $\lambda < 1$,

is $$\Theta^{2n} 0 \,.\, \Theta(Mu, \lambda) = \Theta(0, \lambda)\, \Theta^{2n} u \operatorname{dn} u \,.\, \Pi \left\{ 1 - k^2 \operatorname{sn}^2 \frac{sK}{n} \operatorname{sn}^2 u \right\}$$

(see *Proc. Lond. Math. Soc.*, Vol. XVI., p. 103).

From this is deduced

$$\Theta^{2n} 0 \,.\, \Theta(Nu, \gamma) = \Theta(0, \gamma)\, \Theta^{2n} u \,.\, \Pi \left\{ 1 - k^2 \operatorname{sn}^2 \left(\frac{2t-1}{2n} iK' \right) \operatorname{sn}^2 u \right\},$$

where s, t are integers from 1 to $n-1$ and 1 to n, as before.

This pair is, in fact, only one from a set of transformations whereof the types must be

$$\Theta^{2n} 0 \,.\, \Theta(M_r u, \lambda_r)$$
$$= \Theta(0, \lambda_r)\, \Theta^{2n} u \operatorname{dn} u \,.\, e^{-\frac{\mu_r u^2}{K^2}}\, \Pi \left\{ 1 - k^2 \operatorname{sn}^2 \frac{gK + 2g'iK'}{n} \operatorname{sn}^2 u \right\}$$

(g, g' integers),

$$^*\Theta^{2n} 0 \,.\, \Theta(N_r u, \gamma_r)$$
$$= \Theta(0, \gamma_r)\, \Theta^{2n} u \,.\, e^{-\frac{\nu_r u^2}{K^2}}\, \Pi \left\{ 1 - k^2 \operatorname{sn}^2 \frac{2mK + (2m'+1)\, iK'}{2n} \operatorname{sn}^2 u \right\}.$$

The μ and ν functions which here present themselves are of the forms

$$\mu_r = \frac{\pi i b M_r K}{2\Lambda_r}, \quad \nu_r = \frac{\pi i b' N_r K}{2\Gamma_r}.$$

---

* The Φ formulæ are similar to the above: *e.g.*, in a real transformation

$$\frac{\Phi(Mu, \lambda)}{\Phi^{2n} u} \div \frac{\Phi(0, \lambda)}{\Phi^{2n} 0} = \frac{\Theta(Mu, \lambda)}{\Theta^{2n} u} \div \frac{\Theta(0, \lambda)}{\Theta^{2n} 0}.$$

(For the definition of the Φ-function see a note by the present writer, *Proc. Royal Soc.*, No. 237, 1885.)

If we put $u = K$, we have

$$\gamma'_r = (k')^{2n} \div \Pi\, \mathrm{dn}^4 \frac{2mK + (2m'+1)\, iK'}{2n}.$$

Consequently, by the principle of duality,

$$\lambda_r = k^{2n} \div \Pi\, \mathrm{dn}^4 \left(\frac{2mK' + (2m'+1)\, iK}{2n},\ k'\right).$$

SECT. 8.—*Complete multiplication by* $2n$. *Expression for* $\Theta\, 2nu$.

It is not difficult to show that any two conjugate transformations give complete multiplication; but, as the formulæ are very long to write down, I here notice only a pair which corresponds to the case of λ and γ being each real and <1, viz.,

$$\Theta\,(Mu, \lambda) = C \,.\, \Theta^{2n} u \,\mathrm{dn}\, u \,.\, \Pi \left\{ 1 - k^2 \,\mathrm{sn}^2 \frac{sK}{n} \,\mathrm{sn}^2 u \right\},$$

$$\Theta\,(Nu, \gamma) = C .\, \Theta^{2n} u .\, \Pi \left\{ 1 - k^2 \,\mathrm{sn}^2 \left(\frac{2t-1}{2n} iK'\right) \mathrm{sn}^2 u \right\},$$

where $C \equiv$ a constant, generally.

Changing λ, k, M into k, γ, and $\frac{2n}{N}$, and u into Nu, we have from the former

$$\Theta\, 2nu = C \,.\, \Theta^{2n} (Nu, \gamma)\, \mathrm{dn}\,(Nu, \gamma)\, \Pi \left\{ 1 - \gamma^2 \,\mathrm{sn}^2 \left(\frac{s\Gamma}{n}, \gamma\right) \mathrm{sn}^2 (Nu, \gamma) \right\},$$

and, from Sect. 3,

$$\mathrm{sn}\,(Nu, \gamma) = N \,\mathrm{sn}\, u \,.\, P \div Q, \quad \mathrm{dn}\,(Nu, \gamma) = R \div Q,$$

if

$$\left.\begin{aligned} \Pi\,(1 + \mathrm{ct}^2 u'_0 \,\mathrm{sn}^2 u) &= P \\ \Pi\,(1 + \mathrm{ct}^2 v'_0 \,\mathrm{sn}^2 u) &= Q \\ \Pi\,(1 - \mathrm{dn}^2 v'_0 \,\mathrm{sn}^2 u) &= R \end{aligned}\right\}.$$

(See Notation.)

Hence

$$\Theta\, 2nu = C .\, \Theta^{4n^2} u \,.\, Q^{2n} .\, \frac{R}{Q} \,.\, \Pi \left\{ 1 - \gamma^2 \,\mathrm{sn}^2 \left(\frac{s\Gamma}{n}, \gamma\right) N^2 \frac{P^2}{Q^2} \mathrm{sn}^2 u \right\}$$

$$= C \,.\, \Theta^{4n^2} u \,.\, Q \,.\, R \,.\, \Pi \left\{ Q^2 - N^2 \gamma^2 \,\mathrm{sn}^2 \left(\frac{s\Gamma}{n}, \gamma\right) P^2 \,\mathrm{sn}^2 u \right\},$$

since s is an integer from 1 to $n-1$ inclusively; or, ultimately,

$$\Theta\, 2nu = C\,.\,\Theta^{4n^2}u\,.\,\Pi\left\{1-a^2\,\mathrm{sn}^2\,u\right\},$$

where the number of quadratic factors included in the symbol Π is

$$2n+(n-1)\,2n = 2n^2.$$

Since the function Θ vanishes for any odd multiple of iK', *i.e.*, $\Theta\,(2r+1)\,iK' = 0$, it is seen without difficulty that

$$a \equiv k\,\mathrm{sn}\,\frac{2mK+(2m'+1)\,iK'}{2n},$$

if different integer values be given to m and m' so as to produce the requisite $2n^2$ quadratic factors. In other words, the coefficient a must include all the roots of $\mathrm{sn}\,2nu = \infty$.

The above results, added to those which have been already published by the Society, give a fairly complete development of what I have called the Theory of Composition or Addition. (See *Lond. Math. Soc.*, Vols. XV. and XVI.)

Appendix.

Remarks on the modular equation for an even transformation.

The following remarks are intended to apply more especially to those two cases where an even number is of the forms 2^m and $2\times$ an odd prime number.

Case 1.—$2n = 2^m$.

Here it is convenient to take a relation between λ and k which gives n pairs of inverse moduli $\lambda, \lambda_2, \lambda_3, \lambda_4$, &c., whereof one root only, viz., λ, is real and <1.

For example, when $2n = 2$, then

$$\lambda = \frac{1-k'}{1+k'},\quad \lambda_2 = \frac{1+k'}{1-k'}.$$

When $2n = 4$;

$$\sqrt{\lambda} = \frac{1-\sqrt{k'}}{1+\sqrt{k'}},\quad \sqrt{\lambda_2} = \frac{1+i\sqrt{k'}}{1-i\sqrt{k'}},\quad \sqrt{\lambda_3} = \frac{1+\sqrt{k'}}{1-\sqrt{k'}},$$

and

$$\lambda_4 = \frac{1-i\sqrt{k'}}{1+i\sqrt{k'}}.$$

These correspond to

$$\lambda = k^4 \div \mathrm{dn}^4\left(\frac{iK}{4}, k'\right)\mathrm{dn}^4\left(\frac{3iK}{4}, k'\right),$$

$$\lambda_2 = k^4 \div \mathrm{dn}^4\left(\frac{2K'-iK}{4}, k'\right)\mathrm{dn}^4\left(\frac{2K'+3iK}{4}, k'\right),$$

$$\lambda_3 = k^4 \div \mathrm{dn}^4\left(\frac{4K'-iK}{4}, k'\right)\mathrm{dn}^4\left(\frac{4K'+3iK}{4}, k'\right),$$

$$\lambda_4 = k^4 \div \mathrm{dn}^4\left(\frac{6K'-iK}{4}, k'\right)\mathrm{dn}^4\left(\frac{6K'+3iK}{4}, k'\right);$$

where $\lambda\lambda_3 = 1$, $\lambda_2\lambda_4 = 1$.

Case 2.—*When the order of transformation* $= 2 \times$ *an odd prime number n.*

In this case the modular relation gives another real root, viz., $\lambda_1 < 1$, and we have accordingly what may be called the *second* transformation equation

$$(-)^{\frac{1}{2}(n-1)}\, y = \sin\left\{L - A + B - \ldots + (-)^{\frac{1}{2}(n-1)}(A_1 - B_1 + \ldots)\right\},$$

if $\quad \alpha' = \mathrm{dn}\,\dfrac{2iK'}{n}, \quad \beta' = \mathrm{dn}\,\dfrac{4iK'}{n}, \;\&\mathrm{c.}\ldots, \quad \alpha'\alpha_1' = k' = \beta'\beta_1' = \ldots.$

For example, when $2n = 6$, the equation is

$$y = \sin(-L + A + A_1),$$

where $\quad \alpha' = \mathrm{dn}\,\dfrac{2iK'}{3}, \quad \alpha_1' = \mathrm{dn}\left(K - \dfrac{2iK'}{3}\right).$

Corresponding to λ_1 we have the relations

$$2\Lambda_1 = M_1 K, \quad n\Lambda_1' = M_1 K'$$

Hence the principle of duality gives

$$n\Gamma_1 = N_1 K \;\text{ and }\; 2\Gamma_1' = N_1 K'.$$

(In addition to the two real roots λ, λ_1, there are $n-1$ imaginary ones.)

The following are some formulæ arising from the equation, so far as I have been able to derive them, viz.,

$$y = M_1 x\sqrt{1-x^2}\,\Pi\left\{1-\frac{x^2}{\mathrm{sn}^2\frac{2siK'}{n}}\right\}\left\{1-\frac{x^2}{\mathrm{sn}^2\left(K-\frac{2siK'}{n}\right)}\right\}\div$$

$$\sqrt{1-y^2} = \{1-(1+k')\,x^2\}\,\Pi\left\{1-\frac{x^2}{\mathrm{sn}^2\left(\frac{K}{2}+\frac{2siK'}{n}\right)}\right\}$$

$$\times\left\{1-\frac{x^2}{\mathrm{sn}^2\left(\frac{K}{2}-\frac{2siK'}{n}\right)}\right\}\div$$

$$\sqrt{1-\lambda_1^2y^2} = \{1-(1-k')\,x^2\}\,\Pi\left\{1-k^2\,\mathrm{sn}^2\left(\frac{K}{2}+\frac{2siK'}{n}\right)x^2\right\}$$

$$\times\left\{1-k^2\,\mathrm{sn}^2\left(\frac{K}{2}-\frac{2siK'}{n}\right)x^2\right\}\div$$

$$\text{common denom. } \sqrt{1-k^2x^2}\,\Pi\left\{1-k^2\,\mathrm{sn}^2\frac{2siK'}{n}x^2\right\}$$

$$\times\left\{1-k^2\,\mathrm{sn}^2\left(K-\frac{2siK'}{n}\right)x^2\right\};$$

if $y=\mathrm{sn}\,(M_1u, \lambda_1)$, $x=\mathrm{sn}\,u$, and s is now an integer from 1 to $\frac{1}{2}(n-1)$. (Order of transformation $= 2\times$ an odd prime number n.)

$$\Theta^{2n}0\,.\,\Theta\,(M_1u, \lambda_1) = \Theta\,(0, \lambda_1)\,\Theta^{2n}u\,\mathrm{dn}\,u\,\Pi\left\{1-k^2\,\mathrm{sn}^2\frac{2siK'}{n}\mathrm{sn}^2\,u\right\}$$

$$\times\left\{1-k^2\,\mathrm{sn}^2\left(K-\frac{2siK'}{n}\right)\mathrm{sn}^2\,u\right\},$$

$$\Theta^{2n}0\,.\,\Theta\,(N_1u, \gamma_1) = \Theta\,(0, \gamma_1)\,\Theta^{2n}u\,(1+k\,\mathrm{sn}^2\,u)\,\Pi$$

$$\times\left\{1-k^2\,\mathrm{sn}^2\left(\frac{iK'}{2}\pm\frac{2sK}{n}\right)\mathrm{sn}^2\,u\right\}.$$

$$\lambda_1 = k^{2n}\div(1+k')^2\,\Pi\,\mathrm{dn}^4\left(\frac{iK}{2}\pm\frac{2sK'}{n}, k'\right),$$

$$\gamma_1' = (k')^{2n}\div(1+k)^2\,\Pi\,\mathrm{dn}^4\left(\frac{iK'}{2}\pm\frac{2sK}{n}\right),$$

$$(\Pi\equiv 1, \text{ when } n=1.)$$

$$N_1 = (-)^{\frac{1}{2}(n-1)}(1+k)\,\Pi\,\mathrm{dn}^2\left(\frac{iK'}{2}\pm\frac{2sK}{n}\right)\div\Pi\,\mathrm{dn}^2\frac{2sK}{n}\,\mathrm{dn}^2\left(iK'+\frac{2sK}{n}\right),$$

with a corresponding expression for M_1, obtained by changing the modulus k into k'.

$$\operatorname{sn}(N_1 u, \gamma_1) = N_1 \operatorname{sn} u \, \Pi \left\{ 1 - k^2 \operatorname{sn}^2 \frac{2sK}{n} \operatorname{sn}^2 u \right\}$$

$$\times \left\{ 1 - k^2 \operatorname{sn}^2 \left(iK + \frac{2sK}{n} \right) \operatorname{sn}^2 u \right\}$$

$$\div (1 + kx^2) \, \Pi \left\{ 1 - k^2 \operatorname{sn}^2 \left(\frac{iK'}{2} \pm \frac{2sK}{n} \right) \operatorname{sn}^2 u \right\},$$

where both the signs $\pm$ are to be used.

As I have already observed, the non-real transformations present some difficulties; but the results of this note are, I think, in the main correct. The correlative of the transformation

$$(-)^{\frac{1}{2}(n-1)} y = \sin \{ L - A + B - \ldots \}$$

may be studied in the form

$$(-)^{\frac{1}{2}(n-1)} iz = \tan \{ X_0 - X_1 + X_2 - X_3 + \ldots + (-)^{\frac{1}{2}(n-1)} (Y_1 - Y_2 + Y_3 - \ldots) \},$$

where $$\tan X_0 = i \frac{(1+k)\,x}{1+kx^2}, \quad \tan X_1 = \frac{2ia_1 x}{1+a_1^2 x^2}, \; \ldots\ldots,$$

$$\tan Y_1 = \frac{2ib_1 x}{1+b_1^2 x^2}, \; \ldots\ldots,$$

and the coefficients are

$$a_s = \operatorname{dn} \frac{2sK}{n} \div \operatorname{cn} \frac{2sK}{n}, \quad b_s = k \operatorname{cn} \frac{2sK}{n} \div \operatorname{dn} \frac{2sK}{n},$$

if s be an integer from 1 to $\frac{1}{2}(n-1)$.

When $k = 0$, we have

$$(-)^{\frac{1}{2}(n-1)} i \sin n\theta = \tan \{ X_0 - X_1 + X_2 - \ldots + (-)^{\frac{1}{2}(n-1)} X_{\frac{1}{2}(n-1)} \};$$

if n be an odd prime number, and $\tan X_0 = i \sin \theta$,

$$\tan X_s = 2i \sec \frac{s\pi}{n} \sin \theta \div \left\{ 1 + \sec^2 \frac{s\pi}{n} \sin^2 \theta \right\}.$$

On Unicursal Curves. *By* R. A. ROBERTS.

[*Read Nov.* 12*th*, 1885.]

In this paper I collect together several investigations with regard to unicursal curves, both plane and twisted. Most of the results relate to the curves of the third and fourth degree.

Below is a brief statement of the contents:—

§§ 1—4 relate to certain conditions connecting points on the plane curves.

§§ 5—8 are concerned with conics and quadrics touching the curve of the n^{th} degree in n points.

§§ 9—19 contain miscellaneous properties of the twisted unicursal quartic.

§§ 20—42 contain properties of chords of the same curve.

§§ 43—64 relate to conics and quadrics which divide harmonically chords of curves of the third and fourth order.

§§ 65—68 contain properties of certain twisted curves of the fifth and sixth order.

§§ 69—72 relate to certain unicursal curves which are geodesics on a quadric.

1. In the same way as in § 10 of my "Notes on the Plane Unicursal Quartic" (*Proceedings*, Vol. XVI., p. 50), we get $\frac{1}{2}(n-1)(n-2)$ relations connecting the parameters of the n points where a line meets the general plane unicursal curve of the n^{th} degree. Now, these relations are all linear in the sum, sum of the products in pairs, &c., of the parameters, from which it can readily be inferred that any $n-2$ of these relations are sufficient, and that the remaining $\frac{1}{2}(n-2)(n-3)$ relations can be deduced linearly from the assumed $n-2$.

In finding, then, the relations connecting the points where a curve of the m^{th} degree has p-point contact with the curve at $\frac{mn}{p}$ points, we must modify the resulting conditions by means of the foregoing considerations.

We have seen already, thus, that there are only four systems of conics having quartic contact with the plane unicursal quartic, instead of seven as appears to be the case at first sight (see *Proceedings*, Vol. XVI., p. 50, and Clebsch, *Crelle*, t. 64, p. 64).

In the same way we can show that the number of systems of cubics

having three-point contact with a unicursal quartic at four points is reduced from twenty-six to twenty.

It may also be observed that we can show thus, that there are sixteen conics altogether which touch a unicursal quintic at five points.

2. From the form of the conditions, given *loc. cit.*, that points on the quartic should lie on a line, we can obtain the equation which determines the parameters of the nodes in a certain form.

Let U_1, U_2 be two conditions that four points on the curve should be collinear; then, if the four parameters are supposed to coincide in U_1, U_2, we see, from the equations mentioned above, that

$$U_1+\lambda U_2 = mu^4+nv^4 \qquad (1)$$

where u, v are linear factors corresponding to the parameters of a node. Since, then, the invariant T vanishes for the expression on the right-hand side, we see that λ is determined by the cubic equation found by equating to zero the invariant T of $U_1+\lambda U_2$. Supposing now the expressions to be homogeneous in λ, μ, if

$$\lambda' \frac{d}{d\lambda}+\mu' \frac{d}{d\mu} = \Delta,$$

we have, from (1), $\Delta U_1+\lambda\Delta U_2 = 4mu^3\Delta u \qquad (2),$

taking λ', μ' as a root of v; from which we see that, if we take the combinant Q of the cubics ΔU_1, ΔU_2, we shall obtain a sextic in λ', μ' whose roots are the parameters of the six nodes. Again, it is easy to see that the resultant R of ΔU_1, ΔU_2 gives the points where the six inflexional tangents meet the curve again, and, if we take the combinant P of the same cubics, we shall obtain a covariant quadratic determining two points on the curve. These equations are, then, connected by the relation $R = P^3-27Q$ (Salmon's *Higher Algebra*, Art. 195). In the same way, for the curve of the n^{th} degree, if the conditions for collinear points are

$$U_1=0,\ U_2=0,\ \ldots\ldots\ U_{n-2}=0,$$

we have $\lambda_1 U_1+\lambda_2 U_2+\ldots+\lambda_{n-2} U_{n-2} = \mu u^n+\nu v^n,$

whence $\lambda_1\Delta U_1+\lambda_2\Delta U_2+\ldots+\lambda_{n-2}\Delta U_{n-2} = ku^{n-1}.$

The condition, then, that a linear function of these $n-2$ quantics ΔU_1, &c. of the $(n-1)^{th}$ degree should be a perfect $(n-1)^{th}$ power will give the equation determining the parameters of the nodes.

3. We may notice here a few points in connection with the equa-

tions $\phi(\alpha)-k_1\phi(\alpha')=0$, &c. for the case of the quartic. Taking the three conditions that eight points on the curve should lie on a conic, it is evident that one of the conditions will be identically satisfied if the conic pass through a node; and we have then two relations connecting the remaining six points of intersection of the conic and quartic. Now, suppose the conic to pass through two nodes of the curve, then, for the four other points of intersection, we have

$$\phi(\alpha)-k_1\phi(\alpha')=0,$$

α, α' being the parameters of the node through which the conic does not pass. Thus we see that each of the conditions for the collinearity of four points on the curve expresses by itself that the points lie on a conic through two of the nodes; and, when we take two of these conditions together, it is evident that the conics must break up into a line joining a pair of nodes and another line, and we thus verify the fact that any one of the conditions will necessarily follow from the other two.

We can also show that when one of the conditions is satisfied, say

$$\phi(\alpha)-k_1\phi(\alpha')=0 \quad\text{.............................}(3),$$

the corresponding points will lie on a conic having double contact with a conic of one of the three systems mentioned in §§ 12, 13 of the paper referred to above; for let U be a conic of the system whose points of contact satisfy the equation

$$\psi(\alpha)-k_1\psi(\alpha')=0 \quad\text{.........................}(4),$$

(besides two others), then for any point on the curve we have

$$\sqrt{U} \propto \psi(\vartheta) \quad\text{............................}(5).$$

Hence, writing a conic having double contact with U in the form

$$lx+my+nz+p\sqrt{U}=0 \quad\text{.........................}(6),$$

we have, for four of the points of intersection with the curve,

$$lf_1+mf_2+nf_3+p\psi(\vartheta)\equiv\phi(\vartheta)=0;$$

whence, substituting α and α' successively for ϑ, and recollecting that

$$\frac{f_1}{f_1'}=\frac{f_2}{f_2'}=\frac{f_3}{f_3'}=k_1, \text{ and } \frac{\psi(\alpha)}{\psi(\alpha')}=k_1, \text{ from (4)},$$

we get
$$\phi(\alpha)-k_1\phi(\alpha')=0,$$

which was to be proved. By considering the case when the conic U breaks up into factors, we see that four points connected by the

relation (3) lie on a conic touching the tangents at the node aa', or a conic touching a certain pair of double tangents. We have seen that there is an identical linear relation connecting the quantities $\phi(a) - k_1\phi(a')$, &c. (see § 10 of paper referred to above). Now, if these quantities are connected by another linear relation, in which case, of course, there is only one constant involved, it is easy to see, from what we have shown above, that the four corresponding points are situated on a conic having double contact with an inscribed quadric of the unique symmetrical system (see § 13, *loc. cit.*)

4. It may be observed that we can find, by the method which I have used, the relations connecting the parameters of the points at which the tangents are touched by a curve of the m^{th} class. By considering the reciprocal curve we see that we must have

$$\phi(a) - K_1\phi(a') = 0, \text{ \&c.} \qquad (7),$$

where $$\phi(\vartheta) = (\vartheta - \vartheta_1)(\vartheta - \vartheta_2) \ldots\ldots (\vartheta - \vartheta_p),$$

p being equal to $2m(n-1)$ in general, and a, a' are the parameters of the points of contact of a double tangent. The simplest relations are, however, obtained by considering the cusps of the reciprocal or the points of inflexion of the given curve; in this case the corresponding parameters coincide, and we have then

$$\frac{\phi'(a)}{\phi(a)} = \text{constant, or } \Sigma_p^1 \frac{1}{a-\vartheta} = C_a \qquad (8),$$

where a is the parameter of a point of inflexion. Since there are $3(n-2)$ points of inflexion, we have thus $3(n-2)$ equations of condition, which will always be sufficient in the case of tangents passing through a point; for these tangents are connected by $2n-4$ relations, being $n-2$ less than the number of the points of inflexion. In the case of the conic, however, the relations are not sufficient, as for instance, if the given curve is a quartic, there are seven relations connecting the tangents, and only six are supplied by the conditions (8). For the case of tangents of the cubic touched by a conic, it may be observed that (8) supplies exactly the proper number of conditions.

5. I now proceed to mention a few properties of conics (or quadrics in the case of twisted curves) touching the unicursal curve of the n^{th} degree at n points. Suppose the line joining two points ϑ_1, ϑ_2 on the curve to touch such a conic or quadric S, then we have

$$S_1S_2 - P_{12}^2 = 0 \qquad (9),$$

where $P_{12} = 0$ is the condition that the points should be conjugate with regard to S; but, if $\psi(\vartheta)$ is the quartic of the n^{th} degree which

gives the n points of contact of S, we have $S = \{\psi(\vartheta)\}^2$, from which we see that (9) breaks up into the factors

$$\psi_1\psi_2 \pm P_{12} = 0 \qquad (10).$$

Now, $\psi_1\psi_2 - P_{12}$ is divisible by $(\vartheta_1 - \vartheta_2)^2$, the remaining factor being evidently of the $(n-2)^{\text{th}}$ degree in $\vartheta_1+\vartheta_2$, $\vartheta_1\vartheta_2$, 1.

Hence we see that, if S be a conic having triple contact with the plane unicursal cubic, the tangents to S will meet the curve in points, two of which will belong to a system in involution, as we have seen already (*Proceedings*, Vol. XIV., p. 57).

Again, we can deduce that, if we take a system of points in involution on a twisted cubic, that is, such that the corresponding chords are generators of a quadric containing the curve, then those chords are touched by a quadruply infinite system of quadrics having triple contact with the curve.

In the case of the plane quartic, the relation (10) is of the second degree, and, for one of the three systems alluded to above, breaks up into two linear factors (see *Proceedings*, Vol. XVI., p. 14), thus giving two systems in involution. For the unique system of conics, we may find the relation otherwise, thus: Let u_1, v_1 be linear and u_3, v_3 quantics of the third degree, in the parameter; then, if we write $x = u_1u_3$, $y = v_1v_3$, $z = u_1v_3+v_1u_3$, the conic $S \equiv z^2-4xy = 0$ evidently has quartic contact with the curve, and, it is easy to see, is of the system. Writing, then, a tangent to S in the form $t^2x-tz+y=0$, we have, for the points where it meets the curve,

$$t^2u_1u_3 - t(u_1v_3+v_1u_3)+v_1v_3 = 0,$$

which breaks up into the factors

$$(tu_1-v_1)(tu_3-v_3) = 0.$$

Now, from $tu_3-v_3=0$, we see that each pair of three points where the tangent meets the curve are connected by a relation of the second degree between the sum and product of the parameters. Also, from $tu_1-v_1=0$, we see that the fourth point of intersection is homographically connected with the point of contact on the conic. We shall hereafter consider more particularly the application of the equation (10) to the case of quadrics touching the twisted unicursal quartic at four points.

6. We now proceed to consider the system of conics having double contact with such conics as those which we have been investigating above. Expressing the plane cubic in the form

$$\lambda x = \vartheta, \quad \lambda y = \vartheta^2, \quad \lambda z = 1+\vartheta^3 \qquad (11),$$

if a conic S have triple contact with the curve, the points of contact will be determined by an equation of the form

$$\vartheta^3 - a\vartheta^2 + b\vartheta - 1 \equiv \psi(\vartheta) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

and then, writing the conic V having double contact with S,

$$lx + my + nz + p\sqrt{S} \equiv V = 0,$$

we have, for three of the points where it meets the curve,

$$l\vartheta + m\vartheta^2 + n(1+\vartheta^3) + p(\vartheta^3 - a\vartheta^2 + b\vartheta - 1) = 0 \quad \ldots\ldots\ldots (13),$$

or $$(n+p)\vartheta^3 + (m-pa)\vartheta^2 + (l+pb)\vartheta + n - p = 0,$$

from which it follows that a certain unique conic of the system V can be described through three points on the curve, namely, the conic whose constants l, m, n, p are determined by taking the parameters of the three points as the roots of the equation (13). There are, of course, other conics of the system which are determined by taking two roots of (13) and one root of the same equation with the sign of p changed. It will hence appear that such conics are related to the curve much in the same way as a plane is to a twisted cubic. It is easy to see, then, that if three conics of the system are drawn through the same point of the curve to touch the curve elsewhere, then the curve will be included in the equation

$$\lambda\sqrt{V_1} + \mu\sqrt{V_2} + \nu\sqrt{V_3} = 0.$$

Also, if three conics of the system are drawn to have three point contact with the curve, the curve will be included in the equation

$$\lambda\sqrt[3]{V_1} + \mu\sqrt[3]{V_2} + \nu\sqrt[3]{V_3} = 0.$$

Again, from a property of the twisted cubic we see that, if three conics of the system be drawn through a point P to osculate the curve, their points of contact will lie on a conic of the system passing through P.

It is easy to see that the poles of the chords of contact with S of such osculating conics lie on a unicursal cubic; for, writing (13) in the form $\phi + p\psi$, we have

$$\phi + p\psi = (\vartheta - \vartheta')^3, \text{ identically},$$

whence, if ϑ_1, ϑ_2, ϑ_3 are the roots of $\psi = 0$, we readily obtain

$$(\vartheta_2 - \vartheta_3)\sqrt[3]{\phi_1} + (\vartheta_3 - \vartheta_1)\sqrt[3]{\phi_2} + (\vartheta_1 - \vartheta_2)\sqrt[3]{\phi_3} = 0,$$

which evidently gives the result we have stated.

We can also find conics of the system (V) having double contact

with the curve, that is, by equating to zero the discriminants of both $\phi+p\psi$ and $\phi-p\psi$. If we write the discriminant of $\phi+p\psi$,

$$\Delta p^4+Gp^3+Dp^2+G'p+\Delta'=0,$$

we shall have then $\Delta p^4+Dp^2+\Delta'=0,\ Gp^2+G'=0,$

whence, eliminating p, we obtain

$$\Delta G'^2-DGG'+\Delta' G^2=0,$$

which is of the sixth degree in l, m, n. If S touches the curve at one point, and has four-point contact with it elsewhere, $\Delta'=0$, and the relation connecting l, m, n is of the fifth degree; also, if S has six-point contact with the curve, the relation reduces to the third degree.

Again, since conics having double contact with a fixed conic are equivalent algebraically to circles on a sphere, we can easily deduce that, if conics of the system (V) are drawn through two variable and each of four fixed points on the curve, they will have a constant anharmonic ratio; for such conics will evidently be of the form

$$V+\alpha V'=0,\quad V+\beta V'=0,\quad V+\gamma V'=0,\quad V+\delta V'=0,$$

where α, β, γ, δ are the parameters of the four fixed points; but the anharmonic ratio of these conics is $\dfrac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\delta)(\beta-\gamma)}$, a constant. By the anharmonic ratio of the conics is understood that of their chords of contact with S, or that of their tangents at either of the two points on the curve.

7. I now proceed to consider some properties of conics having quartic contact with a plane unicursal quartic. If the curve be represented by the equations $\lambda x=f_1$, $\lambda y=f_2$, $\lambda z=f_3$, and if f_4 is the biquadratic giving the points of contact of the conic S, we have $\lambda\sqrt{S}=f_4$; hence four of the points where a conic having double contact with S meets the curve are determined by the equation

$$lf_1+mf_2+nf_3+pf_4=0 \quad \text{.......................} (14).$$

We thus see that such conics are related to the curve in the same way as planes are related to a twisted unicursal quartic. There are, then, four conics of the system, apparently, which have four-point contact with the curve.

Since there are three linear relations connecting the sum, sum of the products in pairs, &c. of the roots of f_4 (see *Proceedings*, Vol. XVI., p. 5), we see that f_4 is of the form $u+\lambda v$, where u, v are quantics whose coefficients are constants of the curve, and λ depends upon the

particular conic S. Hence, corresponding to the canonizant of the twisted quartic, we have a quantic of the form $\phi+\lambda\psi$ (*Proceedings*, Vol. XIV., p. 23); and, therefore, corresponding to the invariants S and T of the twisted curve, there are a quadratic, and a cubic in λ respectively. In the case of the conics belonging to one of the three systems, the invariant T of the corresponding twisted curve always vanishes;

for, writing $$x = \alpha\gamma, \quad y = \beta\gamma, \quad z = \gamma^2+\alpha\beta \quad\ldots\ldots(15),$$

where α, β, γ are quadratics in the parameter, $z^2-4xy = 0$, it is easy to see, is a conic of one of the three systems, and the twisted curve $x = \alpha\gamma$, $y = \beta\gamma$, $z = \gamma^2$, $u = \alpha\beta$, evidently has its invariant $T=0$; and then, if S also vanished, the curve should have a cusp. This is not the case for the unique system of symmetrical conics, for which there are two of the system corresponding to the twisted curve whose invariant S vanishes. We see, then, from § 9 of a paper in *Proceedings*, Vol. XIV., p. 22, that conics having double contact with either of two such conics can be described to have double contact with the curve at each pair of the vertices of an infinite number of triangles inscribed in the curve. The tangents to the curve at the vertices of the triangle pass through a point which lies on a conic, as we see by projecting the results in the paper referred to above.

Again, I shall show hereafter (p. 35) that three osculating planes can be drawn to the twisted curve from any point of itself, and that, if the invariant S vanish, the points of contact will lie on a line. Hence, if through any point of the curve three conics of the system which we have just been considering are described to have three-point contact with the curve elsewhere, their points of contact will lie on a line which touches the fixed conic having quartic contact with the curve.

If we equate the invariant T to zero for the twisted curve corresponding to the case of the unique system of conics, we should apparently obtain three conics of the system; but two of these are irrelevant. In fact, it is easy to see that the canonizant of the twisted curve will be of the form

$$l\left\{(\alpha-\vartheta)^4-k_1(\alpha'-\vartheta)^4\right\}+m\left\{(\beta-\vartheta)^4-k_2(\beta'-\vartheta)^4\right\}=0 \quad\ldots(16),$$

where α, α', &c. have the same meaning as before, and $l : m$ is indeterminate. Taking, then, the invariant T of this quantic, l, m are obviously factors, and the corresponding cases are irrelevant. If S be the corresponding conic, we see then that conics having double contact with S can be described to have double contact with the curve at each pair of consecutive vertices of an infinite number of inscribed quadrilaterals.

In the same way, there will be certain definite conics corresponding to numerical values of the absolute invariant $\frac{S^3}{T^2}$ of the twisted curve, and then conics inscribed in the conics so determined can be described to have double contact with the curve at each pair of consecutive vertices of an infinite number of closed polygons inscribed in the plane curve (*loc. cit.*, p. 30). By projection we see that the tangents to the curve at the vertices of the polygon form another polygon inscribed in a unicursal curve of the fourth order.

8. I now proceed to consider some properties of quadrics inscribed in a fixed quadric which touches a unicursal twisted quartic in four points. If the curve be expressed in the same manner as before (*Proceedings*, Vol XIV., p. 23), and if U be the touching quadric, we may evidently write $\sqrt{U}=f_5$, where f_5 is the quantic which determines the four points of contact of U. Let

$$V \equiv lx+my+nz+pu+q\sqrt{U}=0 \qquad (17)$$

be a quadric inscribed in U, then four of the points where it meets the curve will be determined by the equation

$$lf_1+mf_2+nf_3+pf_4+qf_5=0 \qquad (18),$$

and we easily see, thus, that five surfaces of the system can be found such that, for every point of the curve,

$$V_1 : V_2 : V_3 : V_4 : V_5 = \lambda^4 : \lambda^3\mu : \lambda^2\mu^2 : \lambda\mu^3 : \mu^4 \qquad (19).$$

These quadrics are, therefore, related to the general curve in much the same way as spheres are to the circular curve, the imaginary circle at infinity which meets the circular curve four times being replaced by the quadric U having quartic contact with the curve (*loc. cit.*, p. 31). We can then see, in the same way as before, that a quadruply infinite system of quartic surfaces with a double conic can be described through the curve which touches U along its intersection with another quadric.

Again, we see, in the same way, that if four surfaces of the system be described through three variable and four fixed points of the curve respectively, the poles of the plane of the three variable points, which evidently lie on a line, will have a constant anharmonic ratio.

There will evidently, also, be similar theorems involving systems of such quadrics having four-point or ordinary double contact with the curve, as in the case of the circular curve. In this connection it may be noticed, that the result stated in § 14, *loc. cit.*, namely, that the nodal curve of the locus of the osculating circles is a spherical curve of the

sixth order, is incorrect. We may show that the locus is a circular twisted quartic as follows:—From (22), *loc. cit.*, we must take S_1, $4S_2$, $6S_3$, $4S_4$, S_5, respectively proportional to 1, $2p$, p^2+2q, $2pq$, q^2. Substituting, then, these values in the identical relation connecting (20), and dividing by p^2-4q, which belongs to the case of the curve, we get a relation of the second degree connecting p, q, 1, whence these quantities may be expressed as quadratic functions of a parameter. The five spheres S_1, &c. are thus expressed as functions of the fourth degree in a parameter, from which it follows that the locus is a circular twisted quartic.

9. From the property of the unicursal quartic in § 3 of my paper on unicursal quartics referred to above, it may be deduced that the osculating plane at a point P of the curve contains a conic which touches the curve at P, and the sections by the stationary planes. It is easy to see, then, that this conic can be represented by the equations

$$\left.\begin{array}{ll} \lambda x_1 = a_1(\vartheta-\alpha)^2(\phi-\alpha)^2, & \lambda x_2 = a_2(\vartheta-\beta)^2(\phi-\beta)^2 \\ \lambda x_3 = a_3(\vartheta-\gamma)^2(\phi-\gamma)^2, & \lambda x_4 = a_4(\vartheta-\delta)^2(\phi-\delta)^2 \end{array}\right\}\dots\dots\dots(20),$$

where ϕ is the parameter of a variable point on the conic, and ϑ that of the point P on the curve. The tangential equation of this conic is hence found to be

$$\Sigma(\alpha-\beta)^2(\vartheta-\alpha)^2(\vartheta-\beta)^2 a_1a_2\lambda_1\lambda_2 \equiv \Sigma = 0\dots\dots\dots\dots(21);$$

from which it can be seen that the locus of its centre is a unicursal quartic; for the coordinates of the pole of a plane

$$\lambda_1x_1+\lambda_2x_2+\lambda_3x_3+\lambda_4x_4 = 0$$

are the differentials of Σ with regard to λ_1, λ_2, λ_3, λ_4 respectively, and the centre of the conic is the pole of the plane at infinity. From the equations (20), we see that the locus of the conic is the covariant Steiner's quartic which has the given curve for an inflexion curve (*Proceedings*, Vol. xiv., p. 312); the conic is, in fact, a part of the section of this surface by one of its tangent planes, the remaining portion of the section being the other conic which can be described through P to touch the traces of the stationary planes. The other points of intersections of the two conics are, it is easy to see, the points where the plane meets the three double lines of the Steiner's quartic.

From (21), we see that four conics of the system (20) can be described to touch a given plane, and that the parameters of their points of contact are given by the Hessian of the binary quartic which

determines the four points where the given plane meets the curve; for the equation (21) is the Hessian of $\Sigma a_1\lambda_1(\vartheta-a)^4=0$.

The property of the tangent lines of the unicursal quartic referred to above may also be stated as follows:—The planes described through any tangent line and the vertices of the canonical tetrahedron have a constant anharmonic ratio. We can hence derive the theorem that the cone having any point P of the curve as vertex, and containing the tangent QP at P and the lines from P to the vertices of the canonical tetrahedron, has the osculating plane P for the tangent plane along QP.

It may be observed that the planes through the tangent line and the vertices of the tetrahedron of reference have a constant anharmonic ratio for all the curves included in the equations

$$x_1 : x_2 : x_3 : x_4 = (\vartheta-\alpha)^n : (\vartheta-\beta)^n : (\vartheta-\gamma)^2 : (\vartheta-\delta)^n;$$

and there will also exist for these curves conics and cones similar to those we have found above.

10. I now proceed to consider some properties of the osculating planes of the unicursal quartic. By considering the cubic cone which stands on the curve and has any point P on the curve as vertex, we see that three planes can be drawn through P to osculate the curve elsewhere, and that their points of contact lie on a plane passing through P; for these planes are evidently inflexional tangent planes of the cone.

Suppose the canonizant to be written in one of its canonical forms: $\lambda^4+\mu^4+6m\lambda^2\mu^2$, then the condition that four points on the curve should be complanar is

$$1+p_4+mp_2=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22),$$

where $$p_4=\vartheta_1\vartheta_2\vartheta_3\vartheta_4,\quad p_2=\Sigma\vartheta_1\vartheta_2,\quad \text{and}\quad \vartheta=\frac{\lambda}{\mu}.$$

Hence, putting $\vartheta_1=\vartheta_2=\vartheta_3=\vartheta$, say, for an osculating plane, we have

$$1+\vartheta^3\vartheta_4+3m(\vartheta^2+\vartheta\vartheta_4)=0 \;\ldots\ldots\ldots\ldots\ldots\ldots (23);$$

from which we find, if ϑ_1, ϑ_2, ϑ_3 are now the parameters of the points of contact of the osculating planes passing through ϑ_4,

$$\vartheta_1+\vartheta_2+\vartheta_3=-\frac{3m}{\vartheta_4},\quad \vartheta_1\vartheta_2+\vartheta_2\vartheta_3+\vartheta_3\vartheta_1=3m,\quad \vartheta_1\vartheta_2\vartheta_3=-\frac{1}{\vartheta_4} \ldots\ldots (24).$$

Now the coordinates of a plane are, it is easy to see, linear functions of the sum, sum of the products in pairs, &c. of the parameters of the four points in which it meets the curve; thus we see that the plane containing the points considered above (24) passes through a fixed

point and touches a cone of the second order; for its coordinates are quadratic functions of the parameter ϑ_4. The fixed point must evidently be the single contravariant point which the curve has (*Proceedings*, Vol. XIV., p. 311).

If we sought pairs of points A, B on the curve such that the osculating plane at A passed through B, and *vice versâ*, we should have, if x, y were these osculating planes,

$$x = \lambda^3\mu, \quad y = \lambda\mu^3,$$

λ, μ being the parameters of A, B respectively. Hence, since if we substitute $\frac{d}{d\mu}$, $-\frac{d}{d\lambda}$, for λ, μ, respectively, in these expressions, and operate on the canonizant, the result must vanish, we see that the canonizant will not have the terms $\lambda\mu^3$, $\lambda^3\mu$, and therefore be of the canonical form. Hence, from the theory of the binary quartic, we see that there are three pairs of points such as A, B, and that they are determined by the sextic covariant of the canonizant.

Again, from what we have found above, if the curve be written in the form

$$x : y : z : u = \lambda\mu\,(\lambda^2+\mu^2) : \lambda\mu\,(\lambda^2-\mu^2) : \lambda^4-\mu^4 : \lambda^4+\mu^4+6m\lambda^2\mu^2 \ldots (25),$$

we see that six osculating planes of points such as A, B pass through xyz; this point is, therefore, the contravariant point; also the plane u, it can be verified, is the covariant plane (*loc. cit.*, p. 311). Also, from the equations (25), we see that the cone standing on the curve whose vertex is the contravariant point is of the form

$$ay^2z^2+bz^2x^2+cx^2y^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

We see thus that the equations (25) give a unique canonical form of the curve, the point xyz being the contravariant point, the plane u the covariant plane, and the planes x, y, z containing the pairs of the three chords which may be drawn through xyz to the curve. It may be observed that for this form the tetrahedron of reference is the common self-conjugate tetrahedron of all the covariant quadrics of the curve (see *loc. cit.*, p. 310).

If the three points given by the equations (24) lie on a line, they must be co-planar with an arbitrary point on the curve, and therefore, from (22), satisfy the equation

$$1+p_3\vartheta+m\,(\vartheta p_1+p_2) = 0,$$

where ϑ is arbitrary. This becomes, from (24),

$$(1+3m^2)\,(\vartheta-\vartheta_4) = 0,$$

which is identically satisfied if $1+3m^2 \equiv S = 0$, or the invariant S of the curve vanishes. Hence, in this case, we see that the osculating planes at any three collinear points of the quartic meet the curve again at the same point. Hence, if x, y, z are three such osculating planes, and u a plane through their line of contact, the quartic whose invariant S vanishes may be represented by the equations

$$x:y:z:u = (\vartheta-\alpha)^3(\vartheta-\delta) : (\vartheta-\beta)^3(\vartheta-\delta) : (\vartheta-\gamma)^3(\vartheta-\delta)$$
$$: (\vartheta-\alpha)(\vartheta-\beta)(\vartheta-\gamma)(\vartheta-\epsilon) \;\ldots\ldots(27).$$

In this case it is easy to see that the plane through a point P on the curve and the points of contact of the osculating planes from P, touches the quadric containing the curve at its intersection with the covariant plane u.

11. From the fact that the cone standing on the curve whose vertex is the contravariant point is of the form (26), viz., stands on the projection of a lemniscate of Bernoulli, by means of a well-known property of this curve we can derive that the four planes described through the contravariant point C, and a point P of the quartic to touch the curve elsewhere, are an equi-anharmonic system, and their points of contact lie on a plane through C.

Also, from a theorem of mine, given in p. 5 of Vol. XVI. of the *Proceedings*, we find that the six tangent planes of the curve drawn through C and an arbitrary point P in space have their points of contact lying on a cone of the second order whose vertex is C.

From the canonical form (25), we see that the curve will satisfy two equations of the form

$$x^2+y^2+z^2+u^2 = 0, \quad ay^2z^2+bz^2x^2+cx^2y^2 = 0 \;\ldots\ldots\ldots\ldots(28),$$

where

$$a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29).$$

The surfaces (28) subject to the condition (29) will, however, represent two distinct unicursal quartics arising from the two signs of u.

12. Referring now to Salmon's *Surfaces*, Art. 473, Ex. 2, we see that the lines of striction of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

lie on the cone

$$a^6(b^2-c^2)^2y^2z^2+b^6(c^2-a^2)^2z^2x^2+c^6(a^2-b^2)^2x^2y^2 = 0 \;\ldots\ldots(31).$$

But this cone and the hyperboloid are, it is easy to see, connected

together in precisely the same manner as the surfaces (28) and (29). Hence, we see that the lines of striction of a hyperboloid consist of two distinct twisted unicursal quartics; also that the general twisted unicursal quartic may be homographically transformed, so as to become a line of striction of the hyperboloid on which it lies.

It may be observed that the two curves of striction of the hyperboloid (30) may be conjointly represented by the equations

$$x^2 = \frac{a^6}{p^4}\frac{(p^2-b^2)(p^2-c^2)}{(a^2-b^2)(a^2-c^2)}, \quad y^2 = \frac{b^6}{p^4}\frac{(p^2-c^2)(p^2-a^2)}{(b^2-c^2)(b^2-a^2)},$$

$$z^2 = \frac{c^6}{p^4}\frac{(p^2-a^2)(p^2-b^2)}{(c^2-a^2)(c^2-b^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(32),$$

where p is the perpendicular from the centre on the tangent plane to the quadric at the point on the curve. Again, from the equations of the quadric and the cone (31), it is easy to see that we have

$$\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^2 = \frac{x^2}{a^6}+\frac{y^2}{b^6}+\frac{z^2}{c^6} \quad \ldots\ldots\ldots\ldots\ldots(33),$$

from which it can be readily inferred that, if Q be the foot of the perpendicular from the centre on the tangent plane to the quadric at a point P of the curve, then QP is one of the generators at P; for, if we express that QP, whose direction cosines are proportional to

$$x\left(1-\frac{p^2}{a^2}\right), \quad y\left(1-\frac{p^2}{b^2}\right), \quad z\left(1-\frac{p^2}{c^2}\right), \text{ respectively,}$$

is parallel to an edge of the asymptotic cone, we obtain the equation (33).

13. I shall show now that the locus of Q is a twisted unicursal quartic; for it evidently lies on the locus of the feet of the perpendiculars on the tangent planes to the surface, viz.,

$$(x^2+y^2+z^2)^2-(a^2x^2+b^2y^2+c^2z^2)=0\ldots\ldots\ldots\ldots\ldots\ldots(34).$$

But, combining this equation with that of the quadric, we get

$$a^2(b^2-c^2)^2y^2z^2+b^2(c^2-a^2)^2z^2x^2+c^2(a^2-b^2)^2x^2y^2=0\ldots\ldots(35).$$

Hence, since this cone is connected with the quadric in the same manner as the cone (30) and quadric (31), we see that the locus of the feet of the perpendiculars from the centre of a quadric on the generators consists of two twisted unicursal quartics; also, that the general twisted unicursal quartic can be homographically transformed so as to become such a locus on the quadric by which it is contained.

This curve is circular, that is, passes through four points on the imaginary circle at infinity (see *Proceedings*, Vol. XIV., p. 31), and from (35) we see that its inverse with regard to the origin is a curve related in an exactly similar manner to the reciprocal quadric; and, in fact, if ϑ is a parametric angle, we can represent it thus,

$$x : y : z : x^2+y^2+z^2 : 1 = a\,(b^2-c^2)\cos\vartheta : b\,(c^2-a^2)\sin\vartheta$$
$$: c\sqrt{-1}\,(a^2-b^2)\sin\vartheta\cos\vartheta : \left\{b^2\,(a^2-c^2)\sin^2\vartheta + a^2\,(b^2-c^2)\cos^2\vartheta\right\}$$
$$: \left\{(a^2-c^2)\sin^2\vartheta + (b^2-c^2)\cos^2\vartheta\right\} \quad \ldots\ldots\ldots\ldots\ldots(36).$$

We can then easily show that the equation

$$\frac{y^2z^2}{A} + \frac{z^2x^2}{B} + \frac{x^2y^2}{C} - abc\sqrt{-1}\,xyz = 0 \quad \ldots\ldots\ldots\ldots\ldots(37),$$

where $\quad A = \dfrac{1}{b^2} - \dfrac{1}{c^2}, \quad B = \dfrac{1}{c^2} - \dfrac{1}{a^2}, \quad C = \dfrac{1}{a^2} - \dfrac{1}{b^2},$

represents the Steiner's quartic which contains all the foci of the curve (*loc. cit.*, pp. 33, 34), and the intersection of this surface with

$$\frac{y^2z^2}{b^2-c^2} + \frac{z^2x^2}{c^2-a^2} + \frac{x^2y^2}{a^2-b^2} - \sqrt{-1}\,\frac{xyz}{abc}\,(x^2+y^2+z^2) = 0$$

gives the focal curve. The two curves given by (34) and (35) can be conjointly represented by the equations

$$\frac{x^2}{a^2} = \frac{(r^2-b^2)(r^2-c^2)}{(a^2-b^2)(a^2-c^2)}, \quad \frac{y^2}{b^2} = \frac{(r^2-a^2)(r^2-c^2)}{(b^2-a^2)(b^2-c^2)}, \quad \frac{z^2}{c^2} = \frac{(r^2-a^2)(r^2-b^2)}{(c^2-a^2)(c^2-b^2)},$$
$$\ldots\ldots\ldots\ldots\ldots(38),$$

where r is the radius vector to the point on the curve. I now investigate an expression for the arc s of this curve. We have

$$\frac{ds}{dr} = \sqrt{\left\{\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2\right\}}$$
$$= \text{from (38) } r\sqrt{\frac{\{a^2b^2+b^2c^2+c^2a^2-(a^2+b^2+c^2)\,r^2\}}{\{(r^2-a^2)(r^2-b^2)(r^2-c^2)\}}} \quad \ldots\ldots(39).$$

Now, if $d\sigma$ is an element of the arc of the sphero-conic

$$x^2+y^2+z^2 = R^2, \quad \frac{x^2}{\mu^2} + \frac{y^2}{\mu^2-h^2} - \frac{z^2}{k^2-\mu^2} = 0,$$

measured up to its intersection with the confocal

$$\frac{x^2}{\nu^2} - \frac{y^2}{h^2-\nu^2} - \frac{z^2}{k^2-\nu^2} = 0,$$

we can easily show that

$$d\sigma = R\sqrt{\frac{(\mu^2-\nu^2)}{(h^2-\nu^2)(k^2-\nu^2)}}\,d\nu.$$

Now, $d\sigma$ will be equal to ds if

$$h^2 = a^2-b^2,\ k^2 = a^2-c^2,\ \mu^2 = \frac{a^4-b^2c^2}{a^2+b^2+c^2},\ \nu^2 = a^2-r^2,\ R^2 = a^2+b^2+c^2.$$

Thus we see that the arc of the curve is equal to the arc of the sphero-conic which is the intersection of the director sphere and the cone

$$\frac{x^2}{a^4-b^2c^2}+\frac{y^2}{b^4-c^2a^2}+\frac{z^2}{c^4-a^2b^2} = 0,$$

the latter arc being measured up to the intersection with the confocal sphero-conic lying on the cone

$$\frac{x^2}{a^2-r^2}+\frac{y^2}{b^2-r^2}+\frac{z^2}{c^2-r^2} = 0.$$

This representation of the arc is, of course, not possible if the square of the radius of the director sphere is negative. If the radius of the director sphere vanishes, we see, from (39), that the arc is expressible by a pure elliptic integral of the first kind. Putting $c^2 = -(a^2+b^2)$,

we have $$s = \beta F(k,\phi),$$

where $\beta^2 = \dfrac{a^4+b^4+a^2b^2}{2a^2+b^2}$, $k^2 = \dfrac{a^2-b^2}{2a^2+b^2}$, $r^2 = a^2\cos^2\phi+b^2\sin^2\phi$.

14. It may be observed that the equations of the foregoing curves assume a simple form in elliptic coordinates.

Writing the quadric in Cayley's form, viz.,

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1 = 0,$$

and the two confocals

$$\left.\begin{aligned}\frac{x^2}{a+p}+\frac{y^2}{b+p}+\frac{z^2}{c+p}-1 &= 0\\ \frac{x^2}{a+q}+\frac{y^2}{b+q}+\frac{z^2}{c+q}-1 &= 0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(40),$$

the cone (31), being transformed by the substitutions

$$x^2 = \frac{a(a+b)(a+q)}{(a-b)(a-c)}, \text{ \&c.,}$$

gives $\quad q^2(a+p)(b+p)(c+p)-p^2(a+q)(b+q)(c+q) = 0 \text{ ......(41)},$

which must, of course, be divided by $p-q$. From this equation we can derive a property of the curve; for it may evidently be considered as derived from

$$\frac{dp}{\sqrt{\{(a+p)(b+p)(c+p)\}}} \pm \frac{dq}{\sqrt{\{(a+q)(b+q)(c+q)\}}} = 0 \text{...(42)},$$

$$pdq - qdp = 0 \text{..........................(43)}.$$

But (42) is the differential equation of the generators of the surface, and if 2ϑ is the angle between the generators, $\tan^2\vartheta = -\frac{p}{q}$. Thus $\frac{dp}{dq}$ being eliminated between (42) and (43) represents the locus of those points on the generators at which the angle of intersection with the generators of the other system is a minimum.

To obtain the equation of the locus of the foot of the perpendicular from the centre on a generator, we express that the distance of such a point from the centre is a minimum. Hence, since $r^2 = p+q+\text{a}$ constant, we have $dp+dq = 0$, which, combined with (42), gives

$$(a+p)(b+p)(c+p)-(a+q)(b+q)(c+q) = 0 \text{.........(44)};$$

and this, being divided by $p-q$, gives the desired result, which might, of course, have been obtained at once by transforming the equation of the cone (35).

15. It may be added, that more generally the intersection of the quadric (30) and the cone

$$a^2(a^2-k^2)^2(b^2-c^2)^2y^2z^2+b^2(b^2-k^2)^2(c^2-a^2)^2z^2x^2$$
$$+c^2(c^2-k^2)^2(a^2-b^2)^2x^2y^2 = 0 \text{ ......(45)}$$

will break up into two twisted quartics; for this cone satisfies the condition (29). Now, if P be a point on one of the lines of striction, and Q the foot of the perpendicular from the centre on the corresponding generator, and R a point taken on QP, so that

$$\frac{RQ}{PR} = \frac{p^2}{k^2},$$

where p is the perpendicular on the tangent plane at P, we can easily show that the locus of R is one of the curves determined by the cone (45).

Again, if we transform the cone (45) to elliptic coordinates, and put $k^2 = \vartheta$, we have, using Cayley's notation,

$$\frac{(a+p)(b+p)(c+p)}{(p+\vartheta)^2} - \frac{(a+q)(b+q)(c+q)}{(q+\vartheta)^2} = 0 \quad \ldots\ldots(46),$$

which ought to be divided by $p-q$.

Now, this equation evidently arises from (42), and

$$(p+\vartheta)\,dq-(q+\vartheta)\,dp = 0.$$

But, if $2i$ is the angle between the geodesic tangents drawn to the line of curvature $p+\vartheta = 0$, $\tan^2 i = -\left(\frac{p+\vartheta}{q+\vartheta}\right)$.

Hence we see that the curves we are investigating are the loci of points on generators at which a given line of curvature subtends along geodesics a maximum or minimum angle.

16. By generalizing one of the preceding results, we can arrive at the following mode of generation of the twisted unicursal quartic. Let U, V be two quadrics, and O a fixed point; then, if A is a generator of U, and B its conjugate line with regard to V, the plane through O and B will meet A in points lying on a twisted unicursal quartic, the generators such as A being all taken of the same system. Writing $U \equiv xy-zu$, we have $x-\lambda z = 0$, $u-\lambda y = 0$, for a generator A, and the conjugate line of A with regard to V will be then

$$N+\lambda L = 0, \quad M+\lambda P = 0 \ldots\ldots\ldots\ldots\ldots\ldots(47),$$

where L, M, N, P are the four differential coefficients of V. Putting then $\lambda = \frac{x}{z}$ in the equation of the plane passing through the line (47) and the point L', M', N', P', we have

$$(LP'-L'P)\,x^2+(M'N-MN')\,z^2+(LM'+NP'-L'M-N'P)\,xz = 0 \quad \ldots\ldots\ldots\ldots(48),$$

which represents a cubic of which xz is a double line. The remaining intersection with U, it is easy to see, is a twisted unicursal quartic.

This result, it is easy to see, may be also stated as follows:—The locus of points where generators of one system of a quadric are met by homographic tangent planes of a cone of the second degree is a

twisted unicursal quartic. The cone in this case touches the curve in four points; and, since the tangent cone to the containing quadric also touches the curve in the four points lying on its plane of contact, we see that this result is consistent with the theorem given in § (15) of a paper of mine on plane unicursal quartics (*Proceedings*, Vol. XVI., p. 52).

17. We have already considered the case of the curve generated by the locus of the feet of the perpendiculars from the centre of a quadric on the generators of one system. Now, if we consider the locus of the feet of the perpendiculars from an arbitrary point, it is easy to see that the locus must be a circular twisted unicursal quartic; for the feet of the perpendiculars on the tangent planes lie on a nodal cyclide, and the intersection of this cyclide with the quadric must break up into two distinct curves, corresponding to the two systems of generators. We can hence infer that these curves are circular twisted unicursal quartics. Now, if we are given the quadric on which a general twisted unicursal quartic lies, the curve still involves seven arbitrary constants, and four conditions are satisfied by passing through the four points where the quadric is met by the imaginary circle at infinity. Hence we see that circular unicursal quartics lying on a given quadric involve three constants; but the loci considered above involve three constants, namely, the coordinates of the fixed point. Thus we see that these loci are the most general circular twisted quartics. If the quadric containing the curve is one of revolution, the loci will have double contact with the imaginary circle at infinity.

It is evident that the planes through point and the generators touch the tangent cone, and that the planes through point perpendicular to the generators touch the reciprocal of the parallel to the asymptotic cone. Now, these cones are obviously connected by two relations, namely, are such that either has double contact with the reciprocal of the other. Thus we see that the locus of the intersection of mutually rectangular tangent planes to two such cones breaks up into two unicursal cones of the fourth order, namely, the cones standing on the two loci which we have been considering.

Again, considering the case when the point is on the surface, the cone standing on one of the loci is the locus of the intersection of planes drawn through one of the generators at the point with perpendicular planes touching the parallel to the asymptotic cone. Thus we see that, if through any point of the reciprocal sphero-conic arcs be drawn perpendicular to the tangent arcs of a given sphero-conic, the locus of their feet lies on a unicursal cone of the third degree.

We may notice that, in the case of the paraboloid, the locus of the feet of the perpendiculars from a point on the generators consists of two twisted cubics, each passing through two of the points where the surface is met by the imaginary circle at infinity. By inverting the foregoing mode of generation of the circular curve, we see that circles passing through a point which generate a cyclide with the point as cnicnode have their centres on a general circular twisted quartic.

18. We may notice here another mode of obtaining the general curve. Let a twisted cubic referred to four osculating planes be expressed as follows :—

$$x : y : z : u = (\vartheta-\alpha)^3 : (\vartheta-\beta)^3 : (\vartheta-\gamma)^3 : (\vartheta-\delta)^3.$$

Then, for any point on the circumscribed developable, it is easy to see that we may write

$$x : y : z : u = (\phi-\alpha)(\vartheta-\alpha)^2 : (\phi-\beta)(\vartheta-\beta)^2 : (\phi-\gamma)(\vartheta-\gamma)^2 : (\phi-\delta)(\vartheta-\delta)^2 \quad\ldots\ldots\ldots(49),$$

since these are evidently the coordinates of a point lying on the tangent at the point ϑ on the curve. If we express, then, that the point (49) is situated on a quadric, we get an equation of the form

$$A\phi^2+B\phi+C=0 \quad\ldots\ldots\ldots(50),$$

where A, B, C are biquadratics in ϑ. Now, let p, p' be factors of A, and q, q' factors of C; then, if the quadric touches the developable in four points, that is, if the discriminant of (50) is a perfect square, it is easy to see that we must have $B=pq'+p'q$, in which case (50) becomes the product of the factors $p\phi+q$, $p'\phi+q'$. If, then, we take p, q, p', q' as quadratic factors, we see from (49) that the intersection of the developable and the quadric consists of two distinct unicursal quartics which are touched by all the generating planes of the developable; for it is evident that x, y, z, u are any four planes of the system. Thus we see that the unicursal quartic arises from the intersection of a quadric with a quartic developable which touches the surface in four points.

If we take p, q, p', q' as linear and cubic factors respectively, it is evident that the intersection of the quadric and developable consists of a twisted cubic and a unicursal quintic.

19. We can also show that, if a unicursal quartic cone touches a quadric in four points, then its intersection with the quadric breaks up into two twisted unicursal quartics. In order that these curves

should be the most general ones of their kind, it is necessary that the tangent cone to the quadric, which, of course, touches the quartic cone along four edges, should belong to the unique system determined in § (13) of my paper, entitled "Notes on the Plane Unicursal Quartic" (*Proceedings*, Vol. XVI., p. 51); for, if it belonged to one of the other three systems, it is easy to see that a second quadric could be described through either of the curves of intersection.

20. I now proceed to investigate some properties of chords of the unicursal quartic. If we write the canonizant of this curve in the form $\lambda^4+\mu^4+6m\lambda^2\mu^2=0$, the condition that two chords should intersect will be

$$1+q_1q_2+m\,(p_1p_2+q_1+q_2)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(51),$$

where p_1, q_1, p_2, q_2 are the sums and products of the parameters of the extremities of the two chords respectively (*Proceedings*, Vol. XIV., p. 25). It follows, hence, that if two points on the curve belong to a system in involution, then the chord joining them intersects a fixed chord; also, the double points of the involution are the points of contact of the two tangent planes to the curve which pass through the fixed chord. The chord joining the latter pair of points may be called the tangential chord of the fixed chord. Hence we may say, that if two chords intersect, either chord is harmonically connected with the tangential chord of the other,—if by harmonical connection we mean that the corresponding parameters are harmonically connected, in which case it is easy to see that the planes through any three collinear points on the curve and the extremities of the chords form a harmonic system.

21. It is not difficult to see that the locus of chords intersecting a fixed chord is a cubic with a double line; and we can, in fact, obtain the equation of the cubic as follows. Let two planes x, y be drawn through the given chord so as to meet the curve again on chords at the extremities of which the tangents intersect. Only two such planes can be described; for the corresponding chords are evidently determined by the equations

$$1+q^2+m\,(p^2+2q)=0,\quad 1+q_1q+m\,(p_1p+q_1+q)=0,$$

where p_1, q_1 are the coordinates of the given chord. Let z, u, then, be the bitangent planes at the extremities of the chord so determined. We may write now, for any point on the curve,

$$\lambda x=\alpha\gamma,\quad \lambda y=\beta\gamma,\quad \lambda z=\alpha^2,\quad \lambda u=\beta^2 \quad\ldots\ldots\ldots\ldots\ldots(52),$$

where γ is the quadratic which determines the parameters of the extremities of the given chord, and α, β are the quadratics determining those of the chords found above. We then easily find that

the locus of chords intersecting the chord γ is the cubic

$$zy^2 - ux^2 = 0 \quad \text{.............................(53)},$$

of which the given chord xy is a double line.

Now, all the generators of this cubic intersect the line zu; whence we infer that, if any chord of the quartic intersect a given chord, it will also intersect a fixed line lying in two bitangent planes of the curve and completely determined by the given chord.

22. If we take two such cubic loci for two fixed chords, it is evident that they will intersect along the chord of the curve which intersects the two fixed chords. Also, the given curve is common to both loci; hence the remaining part of their intersection is a curve of the fourth order. This curve, it is easy to see, is the locus of points through which two chords drawn to the curve intersect two fixed chords respectively, which is, therefore, a twisted unicursal quartic, as it is easy to see that it must be unicursal.

To find the envelope of the plane of the two chords drawn through the points of the locus to the curve, I observe that the equation of the plane containing the chords p_1, q_1, p_2, q_2 can be written

$$x + y\,(p_1+p_2) + z\,(p_1q_2+p_2q_1) + uq_1q_2 = 0 \quad \text{...........(54)},$$

where x, y, z, u are certain planes, and $1+q_1q_2+m\,(p_1p_2+q_1+q_2) = 0$. In the case we are considering we have $p_1 = \alpha+\beta q_1$, $p_2 = \alpha'+\beta' q_2$, from which we easily find that the plane (54) passes through a fixed point and touches a cone of the second degree.

23. Since the cone standing on the curve whose vertex is a point on the containing quadric has a triple edge, we can infer the following property of the curve from a theorem concerning quartics with a triple point given in the paper of mine referred to above (*Proceedings*, Vol. XVI., p. 57): The planes described through any point P of the containing quadric and the sides of a triangle inscribed in the curve meet the curve again in three chords respectively, which are intersected by a common chord of the quartic.

We can also easily deduce the following theorem from the harmonic properties of a quadrilateral. If six points A, B, C, A', B', C' be taken on the curve so that the planes $A'BC$, $B'CA$, $C'AB$, $A'B'C'$ pass through a point P on the containing quadric, then the chords AA', BB', CC' will be all intersected by a common chord of the curve.

We may also mention the following theorem which can be obtained from the plane cubic (*Proceedings*, Vol. XIV., p. 57), or by means of the surface (53). Let γ be a chord of the curve, and V a cone of the second degree with its vertex on the quartic, which has triple contact

with the curve and touches the two tangents of the curve intersecting γ, then any chord intersecting γ touches V.

24. I now proceed to investigate some properties of closed polygons formed by chords of the curve. We know that the three equations

$$a_1\vartheta_2\vartheta_3+b_1(\vartheta_2+\vartheta_3)+c_1=0,\quad a_2\vartheta_3\vartheta_1+b_2(\vartheta_3+\vartheta_1)+c_2=0,$$
$$a_3\vartheta_1\vartheta_2+b_3(\vartheta_1+\vartheta_2)+c_3=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(55)$$

can coexist for an infinite number of values of ϑ_1, ϑ_2, ϑ_3, provided that the double points of each of the three involutions are harmonically connected. Hence we infer that an infinite number of triangles can be inscribed in the curve so that each side intersects a fixed chord, provided that the tangential chords of each of the three fixed chords are harmonically connected, or, which is the same thing, that each of the fixed chords intersects the tangential chords of the other two.

The same will be true for an infinite number of closed polygons of n sides inscribed in the curve, provided the n fixed chords are connected by three relations. For, eliminating ϑ_2, $\vartheta_3\ldots\vartheta_{n-1}$ between the first $n-1$ equations of the form (55), we obtain a relation of the form

$$A\vartheta_1\vartheta_n+B\vartheta_1+C\vartheta_n+D=0,$$

which can be made to coincide with the n^{th} equation of (55) by the satisfaction of three conditions.

It may be observed that the plane containing two adjacent sides of the polygon touches a developable of the sixth class; for this developable evidently arises from combining the tangential equations of two such cubics as (53); but these surfaces are of the third class and have a line in common, and also, as we have seen, a cone of the second degree; namely, the envelope of the planes containing two chords which intersect two fixed chords respectively. The remaining part of the circumscribed developable is, therefore, as we have stated above, of the sixth class, and also, it is easy to see, unicursal.

25. I now proceed to investigate a method of representing the chords of the quartic by points on a plane. If we put $p=\frac{z}{x}$, $q=\frac{y}{x}$, where p, q are the sum and product, respectively, of the parameters of the extremities of the chord, and x, y, z the coordinates of a point on a plane, it is evident that the chord will be a tangent to the curve if the corresponding point lies on the conic

$$z^2-4xy\equiv U=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(56)$$

and that the points corresponding to two intersecting chords are con-

jugate with respect to the conic

$$x^2+y^2+m(z^2+2xy) \equiv V = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (57),$$

the canonizant being written in the canonical form.

It is easy, to see then, that the points corresponding to chords intersecting on the curve lie on the same tangent of U. Hence, also, the point corresponding to the tangential chord of a given chord P is the pole with regard to U of the polar of P with regard to V.

These two conics are evidently connected by an invariant relation; in fact, the discriminant of $V-\lambda U$ is $4\lambda^3-S\lambda+T=0$, where S and T are the invariants of the curve.

Again, the reciprocal of U with regard to V, viz.,

$$m(x^2+y^2)+(m^2+1)xy-m^2z^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (58)$$

is the locus of points corresponding to chords meeting the curve again, namely, the generators of one system of the containing quadric.

Also, from the invariant relation connecting U and V, the lines divided harmonically by U, V have their poles with regard to U lying on V; from which we see that the problem to circumscribe closed polygons about the curve (*Proceedings*, Vol. XIV., p. 27) is the same as to inscribe polygons in U whose sides should touch the reciprocal of V with respect to U.

26. We can now obtain some results by means of this representation. To a closed polygon inscribed in the curve will evidently correspond a closed polygon circumscribed about the conic U. Hence, from Brianchon's theorem, we can readily deduce that, if a, b, c, d, e, f be six points on the curve, the three chords intersecting the three pairs of chords ab, de; bc, ef; cd, fa, respectively, are all intersected by a common chord of the curve; or, the ruled hyperboloid determined by these three chords meets the curve again in two points lying on the same generator of the other system. Again, corresponding to six tangents a, b, c, d, e, f of the curve, we have six points lying on the conic U; hence, from Pascal's theorem, we can deduce that, if α, α'; β, β'; γ, γ' are the chords intersecting the pairs of tangents ab, de, bc, ef, cd, fa, respectively, then the three chords intersecting the respective pairs α, α', &c., are intersected by a common chord of the curve. We can easily derive several other theorems from the Theory of Plane Conics; as, for instance, the following:—Let a, b, c, d be four points on the curve; then, if α, β are the chords of which ab, cd are the tangential chords, and γ, δ the chords intersecting the pairs of chords bc, ad; ac, bd, respectively, the four chords $\alpha, \beta, \gamma, \delta$ are all intersected by the same chord of the curve. Also, by considering two triangles reciprocal

with regard to the conic V we derive:—If a, b, c, a', b', c' are six chords of the curve such that a' intersects the chords b, c, b' the chords c, a, and c' the chords a, b; then the three chords intersecting the pairs of chords a, a'; b, b'; c, c', respectively, are all intersected by a common chord of the curve. Again, let a, b, c, d be four chords of contact of pairs of intersecting tangents; then, if $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ are the chords intersecting the six pairs ab, cd, ad, bc, bd, ca, respectively, the three chords intersecting the respective pairs $\alpha\alpha', \beta\beta', \gamma\gamma'$ are concurrent. Also, let a, b, c, d be four tangents of the curve, then, if $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ are the chords intersecting the six pairs ab, cd, ad, bc, bd, ca, respectively, the three chords intersecting the respective pairs $\alpha\alpha', \beta\beta', \gamma\gamma'$ are such that each intersects the chords of which the two others are tangentials.

27. From the fact that it is not possible to describe a quadrilateral such that each pair of consecutive vertices should be conjugate with respect to a conic, we infer that it is not possible for four chords of the general curve to form a skew quadrilateral. If, however, the discriminant of V vanish, that is, if the curve have an actual double point, there will be an infinity of such quadrilaterals.

28. Again, suppose a closed "$2n$" gon to be formed by tangents to the curve, then, if u_r, u_{n+r} are the arguments of the parameters corresponding to the r^{th} and $(n+r)^{\text{th}}$ points of contact, we have

$$u_r - u_{n+r} = \tfrac{1}{2}\omega$$

(*Proceedings*, Vol. XIV., p. 27), from which we easily find that the corresponding parameters are harmonically connected with a pair of conjugate roots of the sextic covariant of the canonizant. Hence we infer that such chords intersect one of the three fixed concurrent chords determined by these roots (see § 10).

29. Suppose U and V, referred to their common self-conjugate triangle, to be written in the forms

$$U \equiv x^2+y^2+z^2 = 0, \quad V \equiv ax^2+by^2+cz^2 = 0,$$

we have $$a+b+c = 0 \quad\text{.............................}(59),$$

and then it is readily proved that ax, by, cz, represents the tangential chord of x, y, z. Hence the n^{th} tangential chord is represented by $a^n x, b^n y, c^n z$, from which we easily find that the third tangential chord will always coincide with the given chord if we have

$$ab+bc+ca = 0,$$

or if the invariant S of the curve vanishes.

30. I now proceed to consider some properties of the systems of

chords whose parameters are connected by the relation

$$(a, b, c, f, g, h)\,(p, q, 1)^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(60),$$

that is, whose corresponding points describe a general conic.

I first prove a general property of such systems. Suppose the coordinates of a point on the curve to be expressed in terms of a parameter. Then, if we substitute these expressions in the equation of a quadric S touching the curve in four points, we must get

$$S = \{\phi(\vartheta)\}^2,$$

where $\phi(\vartheta)$ is a binary quartic whose roots give the four points of contact. Now, if the line joining two points touches the quadric S, we have

$$S_1S_2 - P_{12}^2 = 0,$$

where $P_{12} = 0$ is the condition that the two points should be conjugate with regard to S. Hence, if the chord joining the points ϑ_1, ϑ_2 on the curve touch the quadric S, we get a condition which breaks up into the factors

$$\phi(\vartheta_1)\,\phi(\vartheta_2) + P_{12} = 0, \quad \phi(\vartheta_1)\,\phi(\vartheta_2) - P_{12} = 0 \quad\ldots\ldots\ldots(61).$$

Now, the condition must obviously be divisible by $(\vartheta_1 - \vartheta_2)^2$, and as the first factor is not satisfied by $\vartheta_1 = \vartheta_2$, the second factor is divisible by $(\vartheta_1 - \vartheta_2)^2$, and then evidently assumes the form (60). This relation will be sufficiently general to represent any equation of the form (60), because a quadric touching the curve in four points contains five indeterminate constants. We hence infer that chords satisfying a relation of the form (60) are tangents to a quadric touching the curve in four points, or, as we may call it, for brevity, an inscribed quadric. We cannot infer, conversely, that chords touching an inscribed quadric satisfy a relation of the form (60), for there are evidently such chords whose parameters satisfy the first of the factors (61).

31. I now show that chords touching an inscribed quadric are such that the planes described through them and certain four fixed points on the curve have a constant anharmonic ratio. It is easy to see that the plane containing the chords p, q; p', q' is of the form

$$X + p'Y + q'Z = 0,$$

where X, Y, Z contain p, q; but we have also, from (51),

$$1 + mq + mpp' + (m+q)\,q' = 0,$$

and

$$\vartheta^2 - \vartheta p' + q' = 0,$$

where ϑ is the parameter of one of the extremities of p', q'; hence the

plane containing the chord p, q and the point ϑ is

$$\begin{vmatrix} X, & Y, & Z \\ 1+mq, & mp, & m+q \\ \vartheta^2, & -\vartheta, & 1 \end{vmatrix} = 0 \quad\text{.....................(62).}$$

Hence we see that the anharmonic ratio of the planes passing through a chord p, q, and four points ϑ_1, ϑ_2, ϑ_3, ϑ_4 on the curve, is equal to that of the pencil joining the point whose coordinates are

$$1+mq, \quad mp, \quad m+q \text{..........................(63)}$$

to four points lying on the conic U (56). Thus, from the theory of plane conics, we have a general relation of the second degree between the quantities (63), or, which is the same thing, p, q, 1. Also, it is easy to see that the four points ϑ_1, ϑ_2, ϑ_3, ϑ_4 on the curve are the points of contact of the inscribed quadric; from which we can infer that, if chords of the system are such that their tangential chords touch the curve, then the latter lines are the tangents at the points of contact with the quadric. This result may be also stated as follows:— Chords of the curve which are divided in a constant anharmonic ratio by the sides of an inscribed tetrahedron are tangents to a quadric touching the curve at the vertices of the tetrahedron. We may prove this independently thus: If a, b, c, f, g, h are the coordinates of a line with regard to the tetrahedron, it is easy to see that $af : bg : ch$, where $af+bg+ch = 0$, give the anharmonic ratios of the points where the line meets the sides of the tetrahedron. Now, $a = 0$ is the condition that a chord should intersect an edge of the tetrahedron, and $f = 0$ the similar condition for the opposite edge. Hence af is proportional to $RP_{12}P_{34}$, where R is the condition that either extremity of the chord should coincide with one of the vertices of the tetrahedron, and P_{12} is the condition that the chord should intersect the edge (12). Hence, from $af = kbg$, we have

$$P_{12}P_{34} - kP_{13}P_{24} = 0,$$

the factor R dividing out, which is evidently satisfied, as is easily seen from the theory of plane conics.

Since the coordinates of the tangential chords are linear functions of those of a given chord, we see that, if chords of a certain system touch an inscribed quadric, all the successive tangential chords, positive or negative, are tangents to fixed inscribed quadrics.

32. We may now mention a few theorems concerning such systems of chords obtained by means of the representation by points on a plane.

From Pascal's theorem we deduce:—Let a, b, c, d, e, f be any six

chords of the curve touching the same inscribed quadric, and a', b', c', d', e', f' the six chords intersecting the respective pairs ab, de, bc, ef, cd, fa; then the three chords intersecting the pairs $a'b'$, $c'd'$, $e'f'$, respectively, are intersected by a common chord of the curve.

This result may, by McLaurin's theorem for conics, be also stated as follows:—Let α, β, γ be three chords of the curve, and α', β', γ' the chords intersecting the respective pairs $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$; then, if the three latter chords and the chords α, β intersect five fixed chords of the curve respectively, the chord γ will touch a fixed inscribed quadric.

Again, if two variable chords intersect two fixed chords respectively and each other, the third chord drawn through their intersection will touch a fixed inscribed quadric, which, however, is not a general one, as we have evidently only four constants at our disposal.

33. If we express that the plane containing two chords passes through a fixed point, we easily see that the corresponding points on the plane are conjugate with respect to a fixed conic, S say. Hence, if one chord intersect a fixed chord, it is easy to see that the other will satisfy a relation of the form (60). This relation is not, however, the general one, although we have apparently five constants at our disposal; for, if we refer the conics S and V to their common self-conjugate triangle, we find that the locus corresponding to the relation (60) circumscribes the triangle, and therefore satisfies an invariant relation with V. Since there are, then, an infinite number of triangles self-conjugate with regard to V, which are inscribed in the locus, it follows that in the above case three chords of the curve pass through points lying on a curve in space. We can now easily see that the inscribed quadric reduces to a conic meeting the curve in four points; for the quadric will reduce to such a conic if the points of contact are complanar; but to four complanar points on the curve evidently corresponds on a plane a quadrilateral circumscribed about U, whose diagonals are divided harmonically by V; and to the relation (60) corresponds a conic S inscribed in the quadrilateral.

But, when this is the case, it is easy to see that S and V are connected by the invariant relation mentioned above; which is, therefore, the condition that the system of chords corresponding to the equation (60) should intersect a conic meeting the curve in four points.

34. Similarly, if the conic S, in a plane corresponding to the system of chords varying subject to the condition (60), circumscribe triangles circumscribed about the conic U, the inscribed quadric touched by the chords degenerates into a cone. In this case, it is easy to see that we have $$p = \alpha + mr, \quad q = \beta + nr,$$

where p, q, r are the sum, sum of the product in pairs, and product respectively of the parameters ϑ_1, ϑ_2, ϑ_3 of the vertices of the triangle, and α, β, m, n are constants; but the equation of the plane of the triangle is of the form

$$x+ry+\vartheta(z+ru)=0 \quad\text{.......................}(64),$$

where ϑ is the parameter of the point where the plane meets the curve again. Now, for the canonical form of the canonizant, we have

$$1+mq+\vartheta(r+mp)=0 \quad\text{.......................}(65);$$

from which we see that the plane (64) involves a single parameter in the second degree, and, therefore, passes through a fixed point and touches a cone of the second degree.

It is easy to see that there ought to be four cones touching the curve at four given points; for, if S be one of the quadrics touching the curve at these points, and W the quadric containing the curve, we have evidently four cones of the system $S+kW$. Now, one of these cones is that determined by the invariant condition found above, and the three others belong to the case when the expression (60) breaks up into factors, that is, correspond to three pairs of chords of intersection of the conic on the plane with the conic U. These three cones are, it is easy to see, the envelopes of planes containing pairs of intersecting chords which meet the three pairs of chords, respectively, joining the fixed points (see § 22). This result is consistent with some theorems obtained in a paper of mine, entitled "On certain Conics connected with a Plane Unicursal Quartic" (see *Proceedings*, Vol. XVI., p. 14).

35. It may be observed that, in the case of chords intersecting a conic meeting the curve in four points, there is only the relation of the second degree connecting the sum and products of the parameters, the factor of the fourth degree in (61) in these quantities becoming irrelevant, namely, corresponding to the case when the chord passes through one of the four points where the conic meets the curve. This is not the case, however, when the quadric reduces to a cone.

36. We might notice, also, the following theorems. Let a, b, c be three mutually harmonic chords of the curve, and a', b', c' three other chords similarly related to each other, then these six chords are all touched by the same inscribed quadric. Again, if a, b, c are the sides of a triangle inscribed in the curve, and a', b', c' the tangents at the vertices of the triangle, the three chords intersecting the pairs aa', bb', cc' respectively are intersected by a common chord of the curve.

37. Since the tangents of the curve are a particular case of chords satisfying an equation of the form (60), we see that all the tangents are touched by an inscribed quadric. Now I have shown (*Proceedings*, Vol. XIV., p. 309) that the osculating planes, and, therefore, the tangent lines, are touched by a certain contravariant quadric, which quadric, we see now, is inscribed in the curve, the points of contact being the points where certain four tangents meet the curve again (*Proceedings*, Vol. XIV., p. 25). Again, since the chords of contact of intersecting tangents satisfy the equation $V = 0$, we see that these chords are touched by an inscribed quadric. Now, we have seen that the double tangent planes, and, therefore, their chords of contact, touch a contravariant quadric σ' (*loc. cit.*, p. 310). We see, thus, that this quadric is inscribed in the curve; and the points of contact are, it is easy to see, given by the Hessian of the canonizant, namely, the points on the curve at which the tangents meet the curve again (see *loc. cit.*)

We can easily show, hence, that there can be only four independent covariant quadrics. For such a quadric must evidently meet the curve in eight points determined by an equation of the form

$$lC^2 + mCH + nH^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (66),$$

where C is the canonizant, and H its Hessian; whence there are three quadrics corresponding to C^2, CH, H^2, respectively, besides the quadric containing the curve. We thus see that we can determine a doubly infinite system of covariant quadrics which are inscribed in the curve, the points of contact being given by the equation

$$\lambda C + \mu H = 0.$$

It may be observed that the covariant plane meets the curve in the four points given by the equation

$$3TC - 2SH = 0.$$

38. I now proceed to investigate some properties of closed polygons which can be formed by chords of the curve. Since to a closed polygon inscribed in the curve corresponds a closed polygon circumscribed about the conic U in the plane, we see that, if the conic S corresponding to an inscribed quadric satisfy a certain invariant relation, namely, one of the conditions determined by Professor Cayley (*Philosophical Magazine*, Vol. VI., p. 99), then it will be possible to inscribe an infinite number of polygons of a given number of sides in the curve, so that each side will touch the same fixed inscribed quadric.

Again, if a closed polygon of n sides be inscribed in the curve, so

that $n-1$ sides are tangents of a certain system to the same fixed inscribed quadric, the n^{th} side will touch another fixed inscribed quadric. Also, from the theory of plane conics, we can deduce that, if a closed polygon of n sides be inscribed in the curve, so that $n-1$ sides intersect $n-1$ fixed chords of the curve, respectively, then the n^{th} side will touch a fixed inscribed quadric. The conic in the plane corresponding to this latter quadric has, it is easy to see, double contact with U; and hence we can readily show that the chord intersecting the tangents at the extremities of the n^{th} chord touches a quadric having double four-point contact with the curve; for, in that case, the points of contact of the quadric coincide in pairs.

We may also notice the following result. An infinite number of closed polygons can be inscribed in the curve, so that the inverse tangential chord of each side touches a different quadric having double four-point contact with the curve, provided these quadrics are connected by three relations, or, which is the same thing, all but one remain indeterminate.

39. If chords of the curve which touch an inscribed quadric form a closed polygon, it is evident that the corresponding polygon in the plane will be inscribed in a conic S, and be such that each pair of consecutive vertices is conjugate with regard to V, or, which is the same thing, be circumscribed about the covariant harmonic conic of S and V. Hence, corresponding to a certain invariant relation connecting S and V, there are a singly infinite number of closed polygons formed by chords of the curve which touch the same fixed inscribed quadric. We have already considered the case of three chords passing through a point, and have shown that, in that case, the quadric reduces to a conic meeting the curve in four points. We may observe that the polygon cannot be a skew quadrilateral; for we have seen that such a figure cannot be formed by chords of the curve.

Again, if a closed polygon of n sides be formed by chords of the curve, so that $n-1$ sides are tangents of a certain system to a fixed inscribed quadric, then the n^{th} side will touch another fixed inscribed quadric.

40. It may be observed that, in a certain case, that is, if the coefficients in (60) are connected by a certain invariant relation, the corresponding chords will not only touch an inscribed quadric, but also a cone of the second degree, having double contact with the curve, and touching two bitangent planes drawn to the curve through its vertex; for, in a paper published in the *Proceedings*, Vol. XVI., p. 9, I have shown that the tangents to a conic touching two bitangents, and

having double contact with the curve, meet a plane trinodal quartic in two pairs of points whose parameters are connected by the same relation of the form (60). Now, the plane conic has one indeterminate constant, and the position of the vertex of the cone gives three constants, from which we see that (60) must satisfy a single condition in this case. This condition may be easily found by means of the following considerations. Corresponding to the two pairs of points or chords of the curve lying on a tangent plane of the cone, there are, on the plane, two points lying on the conic, S say; these points are conjugate with regard to V, and also with regard to another conic W, because the plane of the chords passes through a fixed point. Hence, writing,

$$V \equiv ax^2+by^2+cz^2 = 0,$$

$$W \equiv a'x^2+b'y^2+c'z^2 = 0,$$

where $$a+b+c = a'+b'+c' = 0,$$

which is evidently allowable, we have, for a pair of points conjugate with regard to both V and W, $xx' = yy' = zz'$. The conic S must then be such that both these points lie on it, and, therefore, it is easy to see, must be of the form

$$S \equiv z^2+kz\,(x \pm y) \pm xy = 0.$$

But, forming the invariants of S and V, we find

$$\Theta\Theta' - \Delta\Delta' = 0 \qquad\qquad (67),$$

which is, therefore, the required invariant condition, satisfied by S in this case. Since (67) is the condition that the covariant harmonic conic of S and V should break up into two points, we see that, in the case we are considering, the third chord drawn to the curve through the intersection of two intersecting chords of the system meets one or other of two fixed chords.

41. With reference to the actual locus of the chords satisfying a relation of the form (60), it may be observed that the surface is of the sixth degree, and has the given quartic for a double curve. There is also another double curve, which appears to be of the tenth degree, namely, the locus of points through which two chords of the system pass. In the case when the inscribed quadric degenerates into a conic meeting the curve in four points, the latter double curve will evidently be replaced by the conic, which is then a triple curve on the surface. In the general case, it is easy to see that the plane of the two intersecting chords of the system touches a developable which is the reciprocal of a twisted unicursal quartic.

42. To find the locus of the points of contact of the chords with the inscribed quadric, S say, which they touch, we may write $S=\phi^2$, where ϕ is a biquadratic in the parameter ϑ of a point on the curve. Putting, then, $x=mx_1+nx_2$, $y=$ &c., for any point on the chord joining the points $x_1y_1z_1u_1$, $x_2y_2z_2u_2$ on the curve, and expressing that this point lies on S, we obtain $m\phi_1+n\phi_2=0$, whence we have

$$x=\phi_1x_2-\phi_2x_1,\quad y=\&\text{c.}$$

Now, these expressions, being divided by $\vartheta_1-\vartheta_2$, are of the third degree in $\vartheta_1\vartheta_2$, $\vartheta_1+\vartheta_2$, 1; from which it follows that the locus of the points of contact is a unicursal curve of the sixth degree. Also, it is easy to see, the locus of the fourth harmonic to the point of contact and the points on the curve is a unicursal curve of the eighth degree touching the quadric S in eight points.

43. I now proceed to consider some cases in which it is possible to inscribe an infinite number of polygons in a unicursal curve, so that the sides may be divided harmonically by a fixed conic. If we express that two points on a plane unicursal cubic are conjugate with regard to a fixed conic, we evidently obtain a relation of the third degree between t_1t_2, t_1+t_2, 1, where t_1, t_2 are the parameters of the two points. Now, if two conditions are satisfied, this relation will break up into one of the second degree and another of the first. I consider this case more particularly. By taking ∞, 0 as the values of the parameters corresponding to the double points of the involution determined by the linear factor, that factor assumes the form t_1+t_2. If, now, we refer the curve to the triangle formed by the node and the points whose parameters are ∞, 0, we may write

$$\lambda x=(t-a)(t-b),\quad \lambda y=t\,(t-a)(t-b),\quad \lambda z=t\,(t-c)\;\ldots\ldots(68),$$

where a, b are the parameters of the node, and c that of the point where z meets the curve again. It is easy to see, then, that the points $\pm t$ are conjugate with regard to the conic

$$S\equiv z\,(y-cx)+(a+b)\,z^2+kxy=0\;\ldots\ldots\ldots\ldots\ldots(69),$$

where k is an arbitrary constant.

Now, eliminating t and λ between (68), we get the equation of the curve in the form

$$xy\,\left\{y-cx+(a+b)\,z\right\}=z\,(y^2+abx^2),$$

from which we see that the line

$$P\equiv y-cx+(a+b)\,z=0$$

intersects z on the curve, and meets the curve elsewhere in two points which are harmonically connected with the points, A, B, say, where z meets the curve, and is, therefore, it is easy to see, completely determined when A, B are given. S is then a conic described through two points A, B on the curve, and through the points where the line P, determined in the manner stated above, is met by the lines OA, OB, where O is the node; for S is evidently satisfied by

$$P = 0, \quad xy = 0.$$

Expressing, now, that the points t_1, t_2 on the curve (68) are conjugate with regard to the conic S (69), and dividing by t_1+t_2, we get

$$(t_1-c)(t_2-c)(t_1t_2+ab)+k\,(t_1-a)(t_1-b)(t_2-a)(t_2-b) = 0 \quad \text{...(70)}.$$

Hence, from this relation, we see that the inscription in the curve of polygons, of the kind described above, is the same analytical problem as that of circumscribing polygons about the conic

$$z^2-4xy = 0 \quad \text{.................................(71)},$$

so as to be simultaneously inscribed in

$$(c^2x-cz+y)(y+abx)+k\,(a^2x-az+y)(b^2x-bz+y) = 0 \quad \text{...(72)}.$$

Since this conic (72) passes through the points where the tangent at the point corresponding to c on (71) is met by the tangents at a, b, it is easy to see that, for the cases of the triangle and quadrilateral, the relation (70) becomes irrelevant, as in the latter case it ought evidently to do. For the case of the pentagon we easily find

$$(a+c)(b+c)+2k\,(a+b)^2 = 0 \quad \text{..................(73)}.$$

44. It is not to be inferred, however, from the above, that there are no conics which have an infinite number of self-conjugate triangles inscribed in the cubic; for we shall show now that such conics arise from the case in which the points A, B, mentioned above, coincide at the node.

Referring the curve to the nodal tangents and the line of inflexions, it may be written $$x^3+y^3-xyz = 0,$$

and may then be represented thus:

$$\lambda x = t, \quad \lambda y = t^2, \quad \lambda z = 1+t^3.$$

If we now express that two points t_1, t_2 on the curve are conjugate with regard to the conic whose equation is

$$a\,(x^2+yz)+b\,(y^2+zx)+cxy = 0 \quad \text{..................(74)},$$

we find that the result is divisible by t_1+t_2, the remaining factor being

$$a(p+q^2)+b(1+pq)+cq=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (75),$$

where $\qquad t_1+t_2=p, \quad t_1t_2=q;$

and, as this relation may be considered as arising from

$$a(t_1t_2+t_2t_3+t_3t_1)+b(t_1+t_2+t_3)+c=0, \quad t_1t_2t_3=1 \ldots\ldots (76),$$

we see that there are an infinite number of triangles in this case. This result coincides with what we know otherwise, for the relation $t_1+t_2=0$ expresses that t_1, t_2 are corresponding points on the curve, and corresponding points are conjugate with respect to all the polar conics of the cubic of which the given curve is Hessian. But the conics (74) are polar conics of

$$x^3+y^3+3xyz=0,$$

which has the given curve for its Hessian. Also, the triangle formed by the points corresponding to those where the Hessian is met by a line, is self-conjugate with regard to the polar conic of any point on the line (see Salmon's *Higher Plane Curves*, Art. 239), which, it is easy to see, is the result represented by the equation (76).

45. In the case of the cuspidal cubic, we put $a=b$ in (68), and if the curve be then transformed to the form

$$y^3-x^2z=0,$$

we find that the conic (69) may be written

$$\{\alpha\beta x-(\alpha^2-\beta)y+z\}\{\alpha(\alpha^2-2\beta)x-(\alpha^2+2\beta)y+2z\}$$
$$+k(y^2-\alpha xy+\beta y^2)=0,$$

where α, β are the sum and product, respectively, of the parameters of two points on the curve.

It is easy to see that there can be no infinite system of triangles self-conjugate with regard to a conic and inscribed in a cuspidal cubic.

It may be observed that, if we express that two points on the cubic $y^3-x^2z=0$, represented by the equations

$$\lambda x=1, \quad \lambda y=t, \quad \lambda z=t^3,$$

are conjugate with regard to the conic

$$ax^2+by^2+cz^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (77),$$

we obtain
$$a+bq+cq^3=0 \qquad (78),$$

where $t_1t_2=q$; from which we see that the points on the curve belong to one or other of three distinct involutions. It may be deduced from this result, that it is possible to inscribe in the curve an infinite number of closed polygons of $2n$ sides, such that each side may be divided harmonically by a conic belonging to the system (77), provided these conics are connected by the single relation

$$q_1q_3 \dots q_{2n-1} = q_2q_4 \dots q_{2n},$$

the q corresponding to each conic being determined by the equation (78).

We may also notice the following result. Since the locus of the intersection of the tangents at the points, and the envelope of the chords joining the points, in the preceding case, are both conics having double contact with the curve (see *Proceedings*, Vol. XIV., p. 57), and passing through the cusp; it can easily be shown that rectangular tangent planes to the cone $y^3-kx^2z=0$, the axes x, y, z being rectangular, intersect on one or other of three quadric cones; also, that the envelope of planes containing two mutually rectangular edges is one or other of three quadric cones, all these cones having double contact with the given cone, and containing its cuspidal edge.

46. Proceeding now to the case of plane unicursal curves of the fourth degree, if we express that two points on the curve whose parameters are t_1, t_2, are conjugate with regard to a conic, we obtain a relation of the fourth degree between t_1+t_2, t_1t_2, 1; and it is evident that, if three conditions are satisfied, this relation will break up into two factors of the first and third degrees respectively. Hence it appears that, if we take a system of pairs of points on the curve whose parameters belong to a given system in involution, then each pair will be conjugate with regard to a conic which is completely determined. Referring the curve to the triangle formed by the nodes, any point on it can be expressed thus:

$$\lambda x = qr, \quad \lambda y = rp, \quad \lambda z = pq \qquad (79),$$

where $p=(t-\alpha)(t-\alpha')$, $q=(t-\beta)(t-\beta')$, $r=(t-\gamma)(t-\gamma')$.

Supposing, then, ∞, 0 as the parameters of the double points of the involution, and $(a, b, c, f, g, h)(xyz)^2=0$ as the equation of the conic, we express that the polar of the node yz, namely,

$$ax+hy+gz=0,$$

passes through the points $-a$, $-a'$, when we find a proportional to

$$2aa'(a+a')(\beta\beta'-\gamma\gamma')A,$$

where
$$A = \begin{vmatrix} 1 & -(a+a'), & aa' \\ 1, & \beta+\beta', & \beta\beta' \\ 1, & \gamma+\gamma', & \gamma\gamma' \end{vmatrix},$$

and h proportional to

$$(\beta\beta'-\gamma\gamma')(\gamma\gamma'-aa')(a+\beta)(a+\beta')(a'+\beta)(a'+\beta');$$

whence, by symmetry, we can at once write down all the other coefficients. If we wish to find, now, the conic corresponding to two points on the curve whose parameters are ϑ_1, ϑ_2, we have merely to substitute $\frac{a-\vartheta_1}{a-\vartheta_2}$, $\frac{a'-\vartheta_1}{a'-\vartheta_2}$, &c., for a, a', &c., respectively.

47. For the tricuspidal quartic $a'=a$, $\beta'=\beta$, $\gamma'=\gamma$ and the equation of the conic becomes then

$$8a^3(\beta-\gamma)^2x^2+\ldots+2(a+\beta)^3(\gamma-a)(\beta-\gamma)xy+\ldots=0 \ldots\ldots (80).$$

Hence, by reciprocation, we see that, if two tangents of a unicursal cubic belong to a system in involution, they will be conjugate with regard to a conic which is completely determined. From this result it may be deduced that the tangent planes drawn through any edge of the cone
$$x^3-3xy^2+z\sqrt{3}(x^2+y^2)=0$$
(the axes being rectangular) to the surface are mutually at right angles; for the tangents intersecting on the curve belong to a system in involution, and the conic in this case is the imaginary one at infinity, *i.e.*, $x^2+y^2+z^2=0$.

48. We have seen that, if two points on the curve are conjugate with regard to a conic, the sum and product of their parameters are connected by a relation of the fourth degree. Now, if four conditions are satisfied, this relation will break up into two factors of the second degree, and, therefore, it would appear that the conic could only involve one arbitrary constant. However, we arrive at the conclusion, which it would be difficult to prove directly, that there are no conics that do not break up into two right lines.

Let x, y be two double tangents of the curve, then, if a, β are the quadratics in the parameter which determine the points of contact, we may write $\lambda x = a^2$, $\lambda y = \beta^2$.

Now, if we take two lines harmonically connected with x, y through their intersection,

$$y^2-k^2x^2=0 \qquad\qquad (81),$$

say, then the condition that two points on the curve should be conjugate with regard to these lines breaks up into the factors

$$\beta_1\beta_2 \pm k\alpha_1\alpha_2=0 \qquad\qquad (82).$$

It may hence be deduced that it is possible to inscribe in the curve an infinite number of closed polygons of $2n$ sides, so that each side may be divided harmonically by a pair of lines of the system (81), provided these pairs are connected by the single relation

$$k_1k_3 \ldots k_{2n-1} \pm k_2k_4 \ldots k_{2n}=0.$$

It may be observed that, in the case also of points on the curve conjugate with regard to two double tangents, the condition breaks up into two factors of the second degree in the sum and product of the parameters, namely,

$$\alpha_1\beta_2 \pm \sqrt{-1}\,\alpha_2\beta_1=0,$$

which are evidently always imaginary.

49. Let us consider, now, two lines intersecting on the curve, such that we may write

$$\lambda x=\vartheta(\vartheta-c)(\vartheta^2-a), \quad \lambda y=(\vartheta-c)(\vartheta^2-b) \qquad\qquad (83),$$

in terms of the parameters. It is easy to see what these lines are; for, forming the equation of the tangents to the curve from xy, we get a result of the form $\alpha x^4+\beta y^4+\gamma x^2y^2=0$, showing that the lines x, y are conjugate factors of the sextic covariant of the four tangents drawn to the curve from a point on itself, which point evidently remains arbitrary; or the lines x, y involve one arbitrary constant. Expressing now that two points on the curve are conjugate with regard to xy, and dividing by $(\vartheta_1+\vartheta_2)(\vartheta_1-c)(\vartheta_2-c)$, we obtain

$$\vartheta_1^2\vartheta_2^2-b(\vartheta_1^2+\vartheta_2^2)+(b-a)\vartheta_1\vartheta_2+ab=0 \qquad\qquad (84).$$

In this case there does not appear to be any geometrical theorem concerning the inscription of polygons in the curve.

50. Again, let x, y be any two lines passing through one of the nodes, then, if α, β, γ are certain quadratic expressions in the parameter, we may evidently write $\lambda x=\alpha\gamma$, $\lambda y=\beta\gamma$; whence, if we express that two points on the curve are conjugate with regard to xy, we get

$$(\alpha_1\beta_2+\beta_1\alpha_2)\gamma_1\gamma_2=0 \qquad\qquad (85),$$

which breaks up into factors, the only one of which belonging to

variable points on the curve is $\alpha_1\beta_2+\beta_1\alpha_2=0$; but the lines xy contain two constants, or one more than would possibly appear to exist at first sight, as stated above.

It is probable that, if we take the general condition that two points on the curve should be conjugate with regard to a conic, and express that this equation breaks up into two factors of the second degree in the sum and product of the parameters, we should come upon the aggregate of all the preceding cases, in which we have shown that this resolution into factors takes place.

51. I now proceed to the problem of the inscription of polygons in a twisted cubic, such that their sides may be divided harmonically by a fixed quadric. If we express that two points on the cubic are conjugate with regard to a quadric, we shall evidently obtain a general relation of the third degree between the sum and product of the corresponding parameters; for a quadric contains nine arbitrary constants, and the plane curve of the third degree the same number; from which we may infer conversely that, in general, if the sum and product of the parameters of two points on the curve are connected by a relation of the third degree, then the two points are conjugate with regard to a fixed quadric. Suppose

$$\phi(\mu) = \frac{(\mu-a)(\mu-b)(\mu-c)(\mu-d)}{(\mu-a')(\mu-b')(\mu-c')(\mu-d')} \quad \ldots\ldots\ldots\ldots\ldots (86),$$

where μ is the parameter of a point on the curve; then

$$\phi(\mu_1)-\phi(\mu_2)=0,$$

being divisible by $\mu_1-\mu_2$, gives a relation of the third degree connecting $\mu_1+\mu_2$, $\mu_1\mu_2$, 1. Now, the relation (86) is satisfied by $\mu_1=a$, and $\mu_2=b$, c, or d, &c.; also by $\mu_1=a'$, $\mu_2=b'$, c', or d', &c.; from which we infer that the quadric in this case has two self-conjugate tetrahedra inscribed in the curve, namely, those formed by the points whose parameters are a, b, c, d and a', b', c', d', respectively. It might appear that the equation (86) contained eight constants, but it is easy to see that there are only six, as (86) involves the coefficients of the Jacobian of the two binary quartics whose roots are a, b, c, d and a', b', c', d', respectively. Hence we see that, if two tetrahedra be inscribed in the curve, there will be a definite quadric U with regard to which these tetrahedra are self-conjugate, and this quadric U will satisfy three invariant relations with the curve. Suppose we take, now, four points on the curve whose parameters are connected by the relations

$$\phi(\mu_1)=\phi(\mu_2)=\phi(\mu_3)=\phi(\mu_4),$$

which is evidently a legitimate assumption. We can hence infer

that a singly infinite number of tetrahedra can be inscribed in the curve which are self-conjugate with regard to the quadric U. It is easy to see, otherwise, that the quadric U satisfies three conditions; for the result of operating with the tangential equation of U on any three quadrics containing the curve must vanish, the three quadrics being taken so as not to be linearly connected.

52. By reciprocation, we see that a singly infinite number of tetrahedra can be formed by osculating planes of the curve so as to be self-conjugate with regard to a quadric Σ, the quadric Σ being connected with the curve by three relations, and being completely determined when two of the tetrahedra are assigned, which may be done arbitrarily. The locus of the vertices of the tetrahedra is evidently another twisted cubic, namely, the locus of the poles with regard to Σ of the osculating planes of the given curve. Hence the result we have arrived at may be stated thus:—If two tetrahedra are formed by osculating planes of a twisted cubic, then their vertices are situated on another twisted cubic.

53. We might also prove this theorem as follows:—From (86) the parameters of the four planes are the roots of the equation $\phi(\mu)-k=0$, from which it is easily seen that the sum, sum of the products in pairs, &c., of the parameters are connected by three linear relations. But since the sum, &c., of three parameters are proportional to linear functions of the coordinates of the intersection of the three corresponding osculating planes, these three relations are equivalent to

$$P+\mu P'=0, \quad Q+\mu Q'=0, \quad R+\mu R'=0 \qquad \text{..............(87)},$$

where P, P', &c. are planes, and μ is the parameter of the fourth osculating plane.

Now these equations, as is well-known, represent a twisted cubic.

By taking two linear relations connecting the sum, sum of the products in pairs, &c., of four parameters, we see, in the same way that osculating planes of the curve can form a doubly infinite system of tetrahedra, which have their vertices on a ruled quadric. Since this quadric involves but six arbitrary constants it must be connected with the curve by three relations.

54. If the quadric we have called Σ above degenerate into a conic and then coincide with the imaginary circle at infinity Ω, it is evident that the curve must satisfy three relations with Ω. Two of these conditions are equivalent to the curve being osculated by the plane at infinity. In this case there will evidently be an infinite number of

three mutually rectangular osculating planes. Referring the curve to rectangular Cartesian coordinates, we may write

$$x = a\,(\vartheta-\alpha)^3, \quad y = b\,(\vartheta-\beta)^3, \quad z = c\,(\vartheta-\gamma)^3,$$

the axes being three planes of the system, and the equation of an osculating plane is then

$$\frac{x}{a}\left(\frac{\beta-\gamma}{\vartheta-\alpha}\right)+\frac{y}{b}\left(\frac{\gamma-\alpha}{\vartheta-\beta}\right)+\frac{z}{c}\left(\frac{\alpha-\beta}{\vartheta-\gamma}\right)+(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)=0.$$

If three of these planes are mutually rectangular, we have

$$\frac{(\beta-\gamma)^2}{a^2(\alpha-\vartheta_1)(\alpha-\vartheta_2)}+\frac{(\gamma-\alpha)^2}{b^2(\beta-\vartheta_1)(\beta-\vartheta_2)}+\frac{(\alpha-\beta)^2}{c^2(\gamma-\vartheta_1)(\gamma-\vartheta_2)}=0,$$

and two other similar equations, which are, it is easy to see, only equivalent to two, namely,

$$\frac{a^2\phi(\alpha)}{\beta-\gamma}=\frac{b^2\phi(\beta)}{\gamma-\alpha}=\frac{c^2\phi(\gamma)}{\alpha-\beta},$$

where $\phi(t)=(t-\vartheta_1)(t-\vartheta_2)(t-\vartheta_3).$

It is easy to see, then, as we stated above, that there are an infinite number of three mutually rectangular osculating planes, and that their intersection lies on the line

$$\frac{ax}{\beta-\gamma}=\frac{by}{\gamma-\alpha}=\frac{cz}{\alpha-\beta}.$$

Reciprocating this result with regard to a sphere, we find that there is a point on the general twisted cubic, through which it is possible to draw an infinite number of three mutually rectangular chords, and that the plane passing through the extremities of these chords contains a fixed line. There are, in fact, two such points, for the cones of the second degree containing the curve involve a parameter in the second degree, and the condition that a cone should contain three mutually rectangular edges is linear in the coefficients. This condition is identically satisfied if the three points at infinity of the curve are mutually rectangular, and there are then an infinite number of three mutually rectangular chords for every point of the curve. In this case the cubic is such that it passes through the feet of the normals drawn from a point to a quadric.

55. Again, let

$$\phi(\mu)=\frac{\mu^4+a\mu^2+b}{\mu(\mu^2+c)} \quad \text{..........................(88)},$$

and suppose two points on the curve whose parameters are connected by the relation $\phi(\mu_1)+\phi(\mu_2)=0$; then this equation, being divisible by $\mu_1+\mu_2$, gives a relation of the third degree connecting $\mu_1+\mu_2$, $\mu_1\mu_2$, 1; and, therefore, the corresponding points are conjugate with regard to a quadric. By expressing the curve in the form

$$\lambda x=1,\quad \lambda y=\mu,\quad \lambda z=\mu^2,\quad \lambda u=\mu^3 \ldots\ldots\ldots\ldots\ldots\ldots (89),$$

the equation of the quadric is easily found to be

$$V\equiv bcx^2+(ca-b)\,y^2+(a-c)\,z^2+u^2+2bxz+2cyu=0 \ldots\ldots\ldots (90).$$

This quadric involves five arbitrary constants, namely, a, b, c, and the two involved by the points on the curve corresponding to the values ∞, 0 of the parameter μ; it therefore satisfies four relations with the curve.

Now, taking four points on the curve, the equations

$$\left.\begin{array}{ll}\phi(\mu_1)+\phi(\mu_2)=0, & \phi(\mu_2)+\phi(\mu_3)=0\\ \phi(\mu_3)+\phi(\mu_4)=0, & \phi(\mu_4)+\phi(\mu_1)=0\end{array}\right\} \ldots\ldots\ldots\ldots (91)$$

are only equivalent to three, or the points belong to a singly infinite system. Hence we infer that a singly infinite system of skew quadrilaterals can be inscribed in the curve, such that each side is divided harmonically by a quadric of the system V. It may also be shown, in a similar manner, that there are a singly infinite number of hexagons and octagons inscribed in the curve which have their sides divided harmonically by V. In these cases the fourth equation of (91), which results from the other three, is satisfied by the factor $\mu_1+\mu_4=0$, and not the factor of the third degree; and then we shall see, further on, the chords joining the points μ_1, μ_4, &c. are generators of a quadric containing the curve, and are divided harmonically by a whole system of quadrics. It may be observed that the chords joining the vertices μ_1, μ_3, &c. are all divided harmonically by a definite quadric; for, dividing $\phi(\mu_1)-\phi(\mu_3)$ by $\mu_1-\mu_3$, we see that the corresponding points are conjugate with regard to

$$bcx^2-(ca-b)\,y^2+(a-c)\,z^2-u^2+2bxz-2cyu=0 \ldots\ldots\ldots\ldots (92),$$

a quadric which has evidently four generators in common with V.

By reciprocating this last result, we see that an infinite number of quadrilaterals, hexagons, and octagons, formed by lines in two planes, can be described so that each pair of consecutive planes containing adjacent sides is conjugate with regard to a quadric which satisfies four relations with the curve.

56. I now proceed to the case in which the condition that two points on the curve should be conjugate with regard to a quadric breaks up into two factors, one linear and the other of the second degree in the sum and product of the parameters. In order that this should be the case, it is evident that the quadric must satisfy two relations with the curve. Taking the double points of the involution determined by the linear factor as ∞, 0, and expressing the curve as at (89), we easily find that the condition that two points on the curve should be conjugate with regard to the quadric

$$S_1 \equiv a\,(y^2+zx)+b\,(z^2+yu)+cxy+dyz+exu+fzu = 0 \ldots\ldots (93)$$

is divisible by $\mu_1+\mu_2$, the remaining factor being

$$ep^2+fq^2+bpq+ap+(d-3e)\,q+c = 0 \ldots\ldots\ldots\ldots\ldots (94),$$

where $$\mu_1+\mu_2 = p, \quad \mu_1\mu_2 = q.$$

It will follow, hence, that pairs of points on the curve belonging to a system in involution, or, which is the same thing, such that the chords joining them are generators of a quadric containing the curve, are conjugate with regard to a system of quadrics involving five absolute constants linearly; also that, if the sum and product of the parameters of two points on the curve are connected by a relation of the second degree, (94) say, the corresponding points are conjugate with regard to the system of quadrics

$$S \equiv lS_1+mS_2+nS_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (95),$$

where S_1 is the quadric given above, and S_2, S_3 are the cones

$$\left.\begin{aligned} S_2 &\equiv cx^2+(d-e)\,y^2+fz^2+2byz+2ezx+2axy = 0 \\ S_3 &\equiv ey^2+(d-e)\,z^2+fu^2+2eyu+2bzu+2ayz = 0 \end{aligned}\right\} \ldots\ldots (96).$$

The equation (95) is found by multiplying (94) by $lp+m+nq$, and then forming the equation of the quadric with regard to which the points μ_1, μ_2 are conjugate. The quadrics of the system (95) have evidently eight points in common, of which four are situated on the curve and are determined by the equation

$$f\mu^4+2b\mu^3+(d+e)\,\mu^2+2a\mu+c = 0.$$

57. It may be observed, that chords satisfying an equation of the form (94) are generators of a definite quartic surface of which the given curve is a double line; also, that these chords touch a singly infinite system of cones of the second degree having their vertices on

the curve, which readily follows from the theory of plane conics, these cones evidently enveloping the quartic surface.

Again, we can see that these chords intersect a singly infinite system of conics meeting the curve in two points; for two chords of the system can be drawn through any point of the curve, and the plane of these chords must evidently meet the quartic surface again in a conic which passes through the extremities of the chords, the quartic being thus generated as the locus of a system of conics.

58. From the theory, then, of the relation (94), we can deduce the following result:—If a quadric S of the system (96) satisfy a certain invariant condition with the curve, it will be possible to inscribe in the curve a singly infinite system of closed polygons of a given number of sides, so that each side may be divided harmonically by the quadric S. For the case of the triangle, it may be observed that the plane of the triangle passes through a fixed line; and, for the case of the quadrilateral, that the diagonals are generators of a quadric containing the curve.

By reciprocation we can arrive at the following result: that there exist a system of quadrics Σ satisfying two relations with the curve, such that, when a third condition is satisfied, it will be possible to form a singly infinite number of closed polygons of lines in two planes, which polygons have the consecutive planes of adjacent sides conjugate with regard to Σ.

59. I now proceed to mention a case in which chords of the curve divided harmonically by a quadric are tangents to another quadric. The curve being expressed in the form (89), it is easy to see that

$$V \equiv a\,(y^2-zx)+b\,(yz-xu)+c\,(z^2-yu)+(lx+my+nz+pu)^2=0 \qquad \text{..............(97)}$$

represents any quadric having triple contact with the curve. Expressing, now, that the line joining two points on the curve touches V,

we get $$V_1V_2-P^2=0 \qquad \text{..............................(98)},$$

where P is the condition that the two points should be conjugate with regard to V. But $$V_1=L_1^2,\quad V_2=L_2^2,$$

and $$2P=-(\mu_1-\mu_2)^2\,\{a+b\,(\mu_1+\mu_2)+c\mu_1\mu_2\}+2L_1L_2,$$

where $$L\equiv l+m\mu+n\mu^2+p\mu^3;$$

hence the equation (98) breaks up into the factors

$$2L_1L_2-(\mu_1-\mu_2)^2\{a+b(\mu_1+\mu_2)+c\mu_1\mu_2\}\pm 2L_1L_2=0.$$

Hence, rejecting the factor

$$a+b(\mu_1+\mu_2)+c\mu_1\mu_2=0,$$

we have $$4L_1L_2-(\mu_1-\mu_2)^2\{a+b(\mu_1+\mu_2)+c\mu_1\mu_2\}=0\;\ldots\ldots\ldots(99),$$

which, it is easy to see, is the condition that the two points should be conjugate with regard to the quadric

$$a(y^2-zx)+b(yz-xu)+c(z^2-yu)+2(lx+my+nz+pu)^2=0\ldots(100),$$

which evidently represents a quadric having triple contact with the curve at the same points as V has, and having conic contact with V along the section by the plane of contact with the curve.

60. I now proceed to consider the twisted unicursal quartic. In this case the condition that two points on the curve should be conjugate with regard to a quadric, is of the fourth degree in the sum and product of the parameters; and it is evident that, if three relations are satisfied, this condition will break up into two factors of the first and third degrees respectively.

Let ∞, 0 be the values of the parameters of the double points of the involution, A, B say, determined by the linear factor, and ϕ, ψ two cubics in the parameter, each of which determines three linear points on the curve; then the curve may evidently be expressed as follows:

$$\lambda x=\vartheta\phi,\quad \lambda y=\phi,\quad \lambda z=\vartheta\psi,\quad \lambda u=\psi\;\ldots\ldots\ldots\ldots\ldots(101),$$

in which case $yz-xu=0$ is the quadric containing the curve. Let us consider, now, the quadric

$$yz+xu\equiv V=0.$$

Expressing, then, that two points on the curve are conjugate with regard to this quadric V, we get

$$(\vartheta_1+\vartheta_2)(\phi_1\psi_2+\phi_2\psi_1)=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(102).$$

Now, a quadric such as V involves four absolute constants implicitly; for it evidently depends upon the position of A, B, and the lines corresponding to ϕ, ψ, each of which is determined by one condition. These constants are separated in the equation (102), for the first factor depends only on the position of A, B, and the second only on that of the two lines. Considering, then, the second factor, which

we can write in the form

$$\frac{\phi_1}{\psi_1}+\frac{\phi_2}{\psi_2}=0 \quad\text{..........................}(103),$$

we can readily infer that there exist a singly infinite system of quadrilaterals and hexagons inscribed in the curve, such that each side is divided harmonically by a quadric V, which, as it involves but four constants, is connected with the curve by five relations. Since, for a diagonal, we have $\phi_1\psi_3-\psi_1\phi_3=0$,

which expresses that the corresponding points are conjugate with regard to $yz-xu=0$, the quadric containing the curve, these diagonals must be generators of that quadric, and, therefore, meet the curve again. The quadrilaterals are then such that their diagonals meet the curve again, and with this condition two quadrilaterals can be assigned arbitrarily, and the corresponding quadrics involve still two undetermined constants. These results, it is easy to see, may be stated as follows:—Let $A_1B_1C_1$, $A_2B_2C_2$ be two triads of collinear points on the curve, then the nine chords A_1A_2, A_1B_2, A_1C_2, B_1B_2, B_1C_2, C_1A_2, C_1B_2, C_1C_2 are divided harmonically by a trebly infinite system of quadrics; also, if $A_3B_3C_3$, $A_4B_4C_4$ are any other pair of triads of collinear points, the nine chords A_3A_4, &c., together with the nine chords mentioned before, are all divided harmonically by a doubly infinite system of quadrics.

61. We can now easily find the general system of quadrics which divide harmonically all the chords of the curve belonging to a system in involution, or, which is the same thing, intersecting a fixed chord of the curve (see § 20).

Let X be the plane drawn through the tangent line at A, so as to meet the curve again on a chord which is harmonically connected with A_1B, and let Y be the plane similarly drawn through the tangent line at B. Also, let Z be the plane drawn through AB, so as to satisfy the same condition; then it is easy to see that we must have

$$\lambda X=\vartheta^4+a\vartheta^2,\quad \lambda Y=\vartheta^2+b,\quad \lambda Z=\vartheta\,(\vartheta^2+c)\,\text{.........}(104);$$

from which it readily follows that, if two points on the curve are conjugate with regard to the pairs of planes XZ or YZ, then the corresponding condition will be divisible by $\vartheta_1+\vartheta_2$. Thus we see that we must add to V the terms $(mX+nY)\,Z$, where m, n are constants; and the resulting quadric, namely,

$$V+(mX+nY)\,Z=0\,\text{......................}(105),$$

evidently contains six arbitrary constants as it ought.

If we express, now, that two points on the curve are conjugate with regard to the quadric (105), and divide by $\vartheta_1+\vartheta_2$, we obtain a relation of the third degree between the sum and product of the parameters, which involves six arbitrary constants. If this relation satisfy three further conditions, it may assume the form

$$\phi(\vartheta_1)-\phi(\vartheta_2)=0;$$

where $\phi(\vartheta)$ has the value (86). In this case, then, there appear to be a singly infinite number of tetrahedra, self-conjugate with regard to a quadric, inscribed in the curve. This quadric involves three arbitrary constants, and, therefore, satisfies six relations with the curve. It also appears that one tetrahedron can be assumed arbitrarily, and that then the corresponding quadric is completely determined.

62. We may now mention the case in which the condition that two points on the curve should be conjugate with regard to a conic, breaks up into two equations of the second degree, in the sum and product of the parameters. In order that this should be the case, it is evident that the quadric must satisfy four relations with the curve, and, therefore, involve five indeterminate constants. It would appear, then, that chords satisfying the general equation

$$(a, b, c, f, g, h)(p, q, 1)^2=0 \qquad (106)$$

are divided harmonically by a definite fixed quadric. All the theorems, then, which we have shown to hold with regard to chords touching an inscribed quadric, will also be true for chords divided harmonically by a certain quadric which satisfies four relations with the curve. This quadric evidently passes through the four points on the curve, the tangents at which are touched by the corresponding inscribed quadric.

The quadric containing the curve is of the system mentioned above, for, if we express that two points on the curve are conjugate with regard to that quadric, the corresponding condition breaks up into the factors corresponding to the tangents of the curve, and to one system of the generators of the quadric, namely, the lines which meet the curve three times.

63. We can verify that the chords of contact of the bitangent planes are divided harmonically by a fixed quadric. Writing the curve in the canonical form (25), we have

$$\lambda x=\vartheta(1+\vartheta^2),\quad \lambda y=\vartheta(1-\vartheta^2),\quad \lambda z=1-\vartheta^4,$$

$$\lambda u=1+\vartheta^4+k\vartheta^2,$$

where
$$\frac{\mu}{\lambda} = \vartheta,$$

and the condition that two tangents at ϑ_1, ϑ_2 should intersect is then

$$k(1+\vartheta_1^2\vartheta_2^2)-2(\vartheta_1^2+\vartheta_2^2+4\vartheta_1\vartheta_2) = 0 \quad\text{..............(107).}$$

Putting, then,
$$\frac{\vartheta_1^2+\vartheta_2^2}{\vartheta_1\vartheta_2} = \rho, \quad \frac{1+\vartheta_1^2\vartheta_2^2}{\vartheta_1\vartheta_2} = \varpi,$$

we have
$$\lambda x_1 x_2 = \varpi+\rho, \quad \lambda y_1 y_2 = \varpi-\rho,$$

$$\lambda z_1 z_2 = \varpi^2-\rho^2, \quad \lambda u_1 u_2 = k^2+k\varpi\rho+\varpi^2+\rho^2-4,$$

and, from (107), ϖ and ρ are connected by the relation

$$2(\rho+4)-k\varpi = 0,$$

from which we evidently obtain a relation of the form

$$ax_1x_2+by_1y_2+cz_1z_2+du_1u_2 = 0 \quad\text{..................(108),}$$

showing that the two points are conjugate with regard to a quadric. This quadric is evidently a covariant, and is of the form

$$V+ks = 0,$$

where V and s are covariant quadrics given in my paper on "Unicursal Twisted Quartics" (*Proceedings*, Vol. XIV., p. 308).

64. When the quadric in the general case reduces to a cone, its vertex will lie on the curve, and the cone then will be determined in the same way as the conic in the case of the plane cubic. If two points on the curve are conjugate with regard to two planes intersecting in a chord of the curve, then the corresponding condition, it is easy to see, breaks up into factors, as is also the case for two planes passing through the intersection of two bitangent planes, and harmonically connected with them. We need not, however, enter into these cases more particularly, for they do not differ essentially from the similar ones for the plane curve.

65. A theorem concerning closed polygons formed by the tangents of the twisted unicursal quartic, which I have given in a paper published in the *Proceedings*, Vol. XIV., p. 26, can be extended to certain twisted unicursal curves of higher orders. Supposing a curve to be expressed by means of the equations

$$x : y : z : u = f_1 : f_2 : f_3 : f_4 \quad\text{........................(109),}$$

where f_1, f_2, f_3, f_4 are quantics of the n^{th} degree in a parameter ϑ, it is easy to see that the coordinates a, b, c, f, g, h of the tangent at the point ϑ are the six Jacobians of the quantics f_1, f_2, &c. Hence, expressing that the tangents at the points ϑ_1, ϑ_2 intersect, by means of the condition

$$af'+fa'+bg'+gb'+ch'+hc'=0 \qquad\qquad (110),$$

we get a relation between ϑ_1, ϑ_2 which is of the degree $2(n-1)$ in each of the variables. Now, this relation must be divisible by a power of $\vartheta_1-\vartheta_2$, and by considering the case of the twisted cubic we see that this factor is $(\vartheta_1-\vartheta_2)^4$. Thus we find that, in general, $2n-6$ tangents of the curve intersect a given tangent. This number will be further reduced if the curve have a cusp; for, if we suppose xyz to be a cusp, f_1, f_2, f_3 must be divisible by a square factor $(\vartheta-a)^2$, say, and then it is evident that the six Jacobians have $\vartheta-a$ as a common factor. Thus we see that, for every cusp the curve has, the coordinates of a tangent are divisible by a linear factor; hence the equation (110) becomes of the degree $2n-6-k$ in each of the variables ϑ_1, ϑ_2, where k is the nnmber of cusps. Let us suppose now $k=2(n-4)$, then we see that the condition that the tangents at the points ϑ_1, ϑ_2 should intersect involves each of these variables in the second degree, and is, therefore, of the same form as in the case of the quartic. To find the limit to the degree of the curve, when it possesses the aforesaid number of cusps, I observe that the circumscribed developable is of the sixth degree (Salmon's *Surfaces*, Art. 361), and that, therefore, the section of this surface by a plane of the system is a curve of the fourth order with $n-3$ cusps; and, since such a curve can at most have 3 cusps, it follows that the greatest value of n is 6. Hence, if a unicursal curve of the fifth degree with two cusps, or a similar curve of the sixth degree with four cusps, satisfy a certain invariant condition, there will be an infinite number of closed polygons of a given number of sides formed by the tangents. It can be shown that it is just possible for the curve to satisfy this condition; for it is easy to see that the general unicursal curve of the n^{th} degree has $4n-15$ absolute invariants, and two conditions are satisfied for every cusp the curve has; hence the curve with $2(n-4)$ cusps has $4n-15-4(n-4)=1$ absolute invariant. This absolute invariant then assumes a numerical value corresponding to the number of the sides of the polygon.

66. Let us consider the case of the quintic. It can be seen withou much difficulty that the curve can be expressed thus:

$$\lambda x = t^2,\ \lambda y = t^3,\ \lambda z = t-a,\ \lambda u = t^5-bt^4 \qquad\qquad (111),$$

where z, u are the osculating planes at the cusps, and x, y are the planes described through one cusp and the tangent line at the other. If, then, a plane meet the curve, we easily find, from (111), that the parameters of the points of intersection are connected by the relations

$$\Sigma_5^1 t = b, \quad \Sigma_5^1 \frac{1}{t} = \frac{1}{a} \quad \ldots\ldots\ldots\ldots (112).$$

Hence, if a plane touch the curve at the points t_1, t_2, and meet it again at the point t_3, we have

$$2(t_1+t_2)+t_3 = b,$$

$$2\left(\frac{1}{t_1}+\frac{1}{t_2}\right)+\frac{1}{t_3} = \frac{1}{a},$$

whence, eliminating t_3, we get

$$4a(t_1+t_2)^2-2t_1t_2(t_1+t_2)-2ab(t_1+t_2)+(b-a)t_1t_2 = 0.$$

If from this relation we seek the condition that three tangents should pass through a point, as in the case of the quartic, we find $a^2 = 0$; but then the curve evidently reduces to a quartic; so that, for the curve we are considering, three tangents can never pass through a point. If four tangents of the curve can form a quadrilateral, we find $b = 7a$.

67. It may be observed that, if the osculating plane of this curve at the point t meet the curve again in the points t_1, t_2, we have, from (112),

$$t_1+t_2+3t = b, \quad \frac{1}{t_1}+\frac{1}{t_2}+\frac{3}{t} = \frac{1}{a},$$

whence, eliminating t, we have

$$a(t_1+t_2)^2-t_1t_2(t_1+t_2)-ab(t_1+t_2)+(b-9a)t_1t_2 = 0.$$

From the form of this relation we infer that, if an invariant condition be satisfied, it will be possible to inscribe in the curve an infinite number of polygons of a given number of sides, such that each of these sides lies in an osculating plane, or, which is the same thing, touches the developable circumscribed about the curve. The relation between a and b which we obtain for the case of the triangle is irrelevant, so that there is no curve for which such triangles exist. For the case of the quadrilateral we find $b+7a = 0$.

68. Let us proceed, now, to consider the sextic. The curve in this case has four cusps, and, if it be referred to the tetrahedron formed by

these points, can be expressed thus:—

$$x : y : z : u = (t-\alpha)^{-2} : (t-\beta)^{-2} : (t-\gamma)^{-2} : (t-\delta)^{-2} \ldots\ldots (113).$$

From this form it is easily seen that the curve is the reciprocal of the developable circumscribed about the general unicursal quartic. Thus we see that the conditions for the possibility of the circumscription of an infinite number of polygons of a given number of sides are the same as in the case of the quartic, and can be expressed in terms of the invariants S and T of the biquadratic whose roots are α, β, γ, δ, viz., the parameters of the cusps. There is a certain peculiarity when the invariant S vanishes. In this case I have shown that three tangents to the quartic curve pass through every point of a certain conic. Hence for the curve we are considering there are an infinite number of triple tangent planes which pass through a fixed point and touch a cone of the second order. This cone contains the sextic, as we see by reciprocating the theorem that the conic considered above is touched by the osculating planes of the quartic.

69. We know that the unicursal quartic lies on a quadric, and I have shown that its osculating planes are touched by another quadric (*Proceedings*, Vol. XIV., p. 309). This is also the case for the bicuspidal quintic; for, from the equations (112), we easily find that the curve lies on the quadric

$$(y-ax)(y-bx)-uz = 0;$$

and, since the reciprocal of the circumscribed developable is a similar curve (Salmon's *Surfaces*, Art. 385), it follows that the osculating planes are touched by a quadric. Again, by reciprocating the quartic, we see that the sextic (113) lies on a quadric, and has its osculating planes touched by another quadric. Hence these three curves satisfy the common conditions of lying on a quadric and being such that their osculating planes or tangent lines are touched by another quadric. It will follow from this that they are capable of being homographically transformed, so as to become geodesics of the quadrics on which they lie; for, by Chasles' theorem, the tangent lines to a geodesic on a quadric are touched by a confocal, and, in general, any two quadrics can be homographically transformed so as to become confocal. In the case of the curves considered above, it is evident that the two quadrics are connected by two invariant relations; as the curves have only one absolute invariant, and two quadrics in general have three. I proceed to investigate what these relations are for the case of the quartic. In the paper referred to above, I have found the equations of the two

quadrics to be

$$V \equiv x^2+y^2+z^2+u^2-2\left(1+\frac{2l^2}{5mn}\right)(yz+xu)-2\left(1+\frac{2m^2}{5nl}\right)(zx+yu)$$
$$-2\left(1+\frac{2n^2}{5lm}\right)(xy+zu),$$

$$s \equiv x^2+y^2+z^2+u^2-2\left(1-\frac{2l^2}{mn}\right)(yz+xu)-2\left(1-\frac{2m^2}{nl}\right)(zx+yu),$$
$$-2\left(1-\frac{2n^2}{lm}\right)(xy+zu),$$

respectively, the curve being referred to the canonical tetrahedron. Now, transforming these equations by putting

$$x+y+z+u = X,\quad x+y-z-u = Y,\quad y+z-x-u = Z,$$
$$z+x-y-u = U,$$

we may write

$$\left.\begin{aligned} V &\equiv 4lmnX^2+l\,(m-n)^2\,Y^2+m\,(n-l)^2\,Z^2+n\,(l-m)^2\,U^2 \\ s &\equiv 2lmnX^2+S\,(lY^2+mZ^2+nU^2) \end{aligned}\right\} \text{...(114)},$$

where $$S \equiv l^2+m^2+n^2.$$

Hence, if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the roots of the discriminant of $V+\lambda s$, we have

$$\lambda_1 : \lambda_2 : \lambda_3 : \lambda_4 = 2S : (m-n)^2 : (n-l)^2 : (l-m)^2 \text{.........(115)}.$$

Thus, remembering that $l+m+n = 0$, we get

$$\lambda_2+\lambda_3+\lambda_4 = \tfrac{3}{2}\lambda_1,\quad \sqrt{\lambda_2}+\sqrt{\lambda_3}+\sqrt{\lambda_4} = 0 \text{..........(116)},$$

which give the two invariant relations connecting V and s.

Let us now suppose the quadrics to be confocal, and, being referred to their principal axes, to be written

$$\left.\begin{aligned} V &\equiv \frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1 = 0 \\ s &\equiv \frac{x^2}{a+p}+\frac{y^2}{b+p}+\frac{z^2}{c+p}-1 = 0 \end{aligned}\right\} \text{..............(117)};$$

then, if the plane X coincide with the plane at infinity, we have, from the invariant relations given above,

$$3+2p\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 0,$$

$$\left(1+\frac{p}{a}\right)^{\frac{1}{2}}+\left(1+\frac{p}{b}\right)^{\frac{1}{2}}+\left(1+\frac{p}{c}\right)^{\frac{1}{2}}=0;$$

whence, eliminating p and clearing of radicals, we get a relation which is equivalent to

$$(bc)^{\frac{1}{2}}+(ca)^{\frac{1}{2}}+(ab)^{\frac{1}{2}}=0 \quad \dots\dots\dots\dots\dots\dots(118).$$

Now, if the curve is real, the quadric must be a ruled surface, and, therefore, one of the quantities a, b, c negative; but this cannot obtain with the relation (118), from which we infer that the curve in this case is always imaginary. If one of the planes Y, Z, U, in equations (114), coincides with the plane at infinity, the invariant relations (116) give $\quad 3abc-p\,(3ab-2ca-2cb)=0,$

$$\left(1+\frac{p}{a}\right)^{\frac{1}{2}}+\left(1+\frac{p}{b}\right)^{\frac{1}{2}}-1=0,$$

whence, eliminating p, we get

$$3a^2b^2-2abc\,(a+b)-c^2\,(a-b)^2=0 \quad \dots\dots\dots\dots(119),$$

a relation which it is possible to satisfy with any one of the quantities a, b, c, negative, and the other two positive. Hence, if the axes of a gauche hyperboloid are connected by the relation represented by (119), it would appear that there lies on the surface a real special geodesic which is a twisted unicursal curve of the fourth order.

If the invariant S of the curve vanishes, the quadrics reduce to a conic, and the tangential equations of V and s may then be written

$$\Sigma \equiv \alpha^2+\beta^2+\gamma^2-\tfrac{3}{4}\delta^2=0,$$

$$\sigma \equiv \alpha^2+\vartheta\beta^2+\vartheta^2\gamma^2=0,$$

where ϑ is a cube root of unity. Hence it appears that, if σ be made to coincide with the imaginary circle at infinity, or a focal conic of Σ, the surface will be always imaginary.

70. If the invariant T vanishes, we know that the quartic lies on a series of quadrics, and the question may be then independently investigated as follows.

It is easily seen, in this case, that the curve may be represented by the equations

$$\lambda x=1, \quad \lambda y=\vartheta^4, \quad \lambda z=\vartheta^2, \quad \lambda u=\vartheta^3+\vartheta \quad \dots\dots\dots(120),$$

from which it is readily found that

$$V \equiv a\,(2z^2+zx+zy-u^2)+b\,(z^2-xy)=0 \quad \dots\dots\dots(121)$$

is any quadric containing the curve, and

$$s \equiv 3z^2 - u^2 + xy = 0 \ldots\ldots\ldots\ldots\ldots\ldots(122)$$

is the quadric touched by the osculating planes. Hence, forming the discriminant of $V + \lambda s$, we get

$$(\lambda + a)(\lambda - b)\left\{3\lambda^2 + 2(a-b)\lambda - (a+b)^2\right\} = 0,$$

from which we see that the roots of this equation are connected by the relations

$$2(\lambda_1 + \lambda_2) - 3(\lambda_3 + \lambda_4) = 0, \quad (\lambda_1 - \lambda_2)^2 + 3\lambda_3\lambda_4 = 0,$$

whence we may derive

$$4(\lambda_1 - \lambda_2)^2 - 3(2\lambda_1 + 2\lambda_2 - \lambda_3 - 3\lambda_4)(3\lambda_3 + \lambda_4 - 2\lambda_1 - 2\lambda_2) = 0 \ldots(123).$$

The latter condition, being applied to the confocal quadrics (117), gives either

$$4c^2(a-b)^2 - 3(2bc + 2ca - ab)(3ab - 2bc - 2ca) = 0 \ldots\ldots(124),$$

or

$$4a^2b^2 - 3(2ab - ca - 3bc)(3ca + bc - 2ab) = 0 \ldots\ldots\ldots(125),$$

the latter of which relations only can be satisfied by a gauche hyperboloid. Hence, if the ruled surface (117) satisfy the relation (125), it will contain a geodesic which is a quadri-quadric curve with a double point. In this and the preceding cases, I have not investigated the conditions satisfied by the axes of the confocal quadric s (114); for it is easy to see that there are geodesics lying on a ruled quadric whose tangents touch confocal quadrics of all the three kinds. The same also holds in all other cases.

71. In a similar way, we should find a relation connecting the axes of the hyperboloid, if there were a geodesic curve of the fifth degree of the form given by the equations (111) lying on the surface; but I do not think it worth while investigating this relation. For the curve of the sixth degree, given by the equations (113), it is easy to see that the relations connecting the quadrics are exactly the same as in the case of the general unicursal quartic; for the curve is the reciprocal of the developable circumscribed about the quartic, and the quadrics containing the curve and touched by the osculating planes are what s and V become, respectively, when we substitute tangential coordinates λ, μ, ν, ρ for X, Y, Z, U in (114). Thus we see that, corresponding to the condition (119) satisfied by the ruled quadric V (117), there lie on the surface distinct geodesic curves of the fourth and sixth degrees respectively.

Again, consider the curve which is the reciprocal of the developable circumscribed about the unicursal quadri-quadric; then, as in the preceding case, the quadrics containing the curve, and touched by the osculating planes, are evidently connected by the same relations as the quadrics (121) and (122). Hence, also, in this case we infer that, corresponding to the condition (123), the ruled quadric will have distinct geodesics of the fourth and sixth degrees respectively.

72. In addition to the preceding results, it may be noticed that in a certain case a twisted cubic may be a geodesic on a ruled quadric; for the osculating planes to such a curve are evidently touched by an infinite number of quadrics. Writing the curve as follows:

$$\lambda x = 1, \quad \lambda y = t, \quad \lambda z = t^2, \quad \lambda u = t^3,$$

the equation $\quad V \equiv ux - yz + a(y^2 - zx) + b(z^2 - yu) = 0 \quad \text{.........(126)}$

represents any quadric containing the curve, and

$$s = ux - 9yz = 0$$

represents one of the quadrics touched by the osculating planes. Forming, then, the discriminant of $V + \lambda s$, we get

$$(9\lambda^2 + 10\lambda + 1 + ab)^2 - 4ab(1+\lambda)^2 = 0,$$

from which we can obtain two relations connecting the roots λ_1, λ_2, λ_3, λ_4. If from these relations we form another involving the differences only of λ_1, &c., we find

$$16(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4) + 9(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)^4 = 0.$$

Applying, then, this condition to the quadric V in the form (117), we

get $\quad 16a^2b^2c^2(c-a)(c-b) + 9(bc + ca - ab)^4 = 0 \quad \text{.........(127)},$

which can be satisfied by a negative value of b, and positive values of a and c. Hence, for the ruled quadric which satisfies the relation (127), there is a geodesic on the surface which is a twisted cubic.

Thus altogether, corresponding to certain conditions satisfied by the axes of a ruled quadric, we have found six distinct algebraic curves which are geodesics on the surface. It would be interesting to verify that these curves, with the corresponding conditions, satisfy the differential equation of the geodesics on a quadric (see Salmon's *Geometry of Three Dimensions*, Art. 427); this would, however, involve a good deal of labour.

On Clifford's Theory of Graphs. *By* A. Buchheim, M.A.

[*Read Nov.* 12*th*, 1885.]

In the present paper I attempt to reconstruct Clifford's theory of Graphs, from the lithographed volume of Mathematical Fragments, and from the letter to Prof. Sylvester, published in the first volume of the *American Journal of Mathematics.*

The first published account of a theory of graphs is contained in Prof. Sylvester's paper "On an Application of the new Atomic Theory to the Graphical Representation of the Invariants and Covariants of Binary Quantics" (*American Journal*, Vol. I., p. 64). In this paper Prof. Sylvester showed how any concomitant of a binary quantic could be graphically represented by a figure entirely analogous to the graphic formulæ used by chemists. In a letter to Prof. Sylvester, printed in the same volume of the *Journal*, Clifford showed that Sylvester's qualitative representation could be made quantitative, inasmuch as the graph could be interpreted as a direction to perform certain multiplications which would result in the form in question. It is this quantitative theory of graphs that I have attempted to explain in this paper. As regards the contents of the paper, I remark that I have given certain preliminary explanations of the elementary processes employed, and have then investigated the theory of the cubic and quartic. The parts of Clifford's Fragments that I have not considered are:—(1) The theory of systems of quantics, where the necessity of distinguishing between different forms makes the use of graphs troublesome, and where very little seems to be gained by using them. (2) The theory of the quintic, where Clifford has treated an unsymmetric graph as if it were symmetrical, and where the correct theory would involve more trouble than it seems worth. (3) A few fragmentary notes, some of which I was unable to understand.

It must be distinctly understood that, excepting a few corrections and the last section (on form-systems), this paper contains nothing that is not explicitly or implicitly contained in Clifford's Fragments,* and that my only object has been to make Clifford's theory more accessible, in the hope that it may be taken up by others, so that it may appear whether the method is likely to lead to new results. I must own that, owing to its essential

* An alteration which I have made in Clifford's method is pointed out and justified below.

identity with the symbolic methods employed by Cayley and Clebsch, it does not seem likely to furnish anything that could not be found quite as easily by the older methods; at the same time, there can, I think, be very little doubt that the representation of a concomitant by a graph throws considerable light on the genesis of a form system, and on Gordan's proof of the existence of a finite form system as presented by Clebsch.

I am not quite sure that I have presented the theory from Clifford's point of view; it is not quite clear what he conceived to be the function of the polar variables in a form, and what relation he supposed the polar form to bear to the ultimate form in which all the variables are scalars. The only passage bearing on this point (*Math. Papers*, p. 256, l. 24) is by no means conclusive, but I imagine that Clifford did not regard the polar form as a blank form to be filled up by multiplication by a polar variable, which is the point of view from which the form is considered in this paper.

I. *Fundamental Operations and Notation.*

If we use a well-known symbolic notation, we can write

$$a_{11}x_1y_1+a_{12}x_1y_2+a_{21}x_2y_1+a_{22}x_2y_2 = (a_1x_1+a_2x_2)(a_1y_1+a_2y_2) = a_xa_y,$$

if we stipulate that $a_ia_k = a_{ik}$; and then, if we assume that the a's are to obey the commutative law, we have

$$a_{12} = a_1a_2 = a_2a_1 = a_{21},$$

that is to say, if we get a lineo-linear form by a symbolic multiplication of two linear forms, the resulting form must be *symmetrical*,* if the coefficients of the linear forms obey the commutative law.

In the same way, if we multiply together any number of linear forms, we get a form linear in the same number of variables, and it is easy to see that, if the coefficients of the linear forms obey the commutative law, the resulting multiple linear form will be symmetrical; that is to say, that all terms with the same number of 1's and the same number of 2's in the subscript indices will have the same coefficient; thus, for a triply linear form, we get a set of terms

$$x_1y_1z_2, \quad x_1y_2z_1, \quad x_2y_1z_1,$$

and, since each of these terms has two 1's and one 2 in the subscript indices, they will all have the same coefficient, $a_1^2a_2$; moreover, we have

$$a_xa_y = a_ya_x,$$

$$a_xa_ya_z = a_xa_za_y = \&c.$$

* As regards interchanges of x_1 with y_1 and x_2 with y_2.

Now, suppose we multiply two linear forms together without making any stipulation as to commutative multiplication, we shall obviously get an unsymmetrical lineo-linear form; for we get

$$(a_1x_1+a_2x_2)(a_1y_1+a_2y_2) = a_1a_1x_1y_1+a_1a_2x_1y_2+a_2a_1x_2y_1+a_2a_2x_2y_2,$$

and, if we do not stipulate that

$$a_1a_2 = a_2a_1,$$

this form is obviously unsymmetrical. In the same way, if we multiply n linear forms together, we shall get an n-tuply linear form, and it is obvious that it will consist of n^2 terms, no two of which will have the same coefficient.

If we suppose all the pairs of variables to become identical, we get a binary quantic of the n^{th} order, in which the coefficient of $x_1^r x_2^s$ will be the sum of all the products of a_1, a_2 containing a_1 r times as a factor, and a_2 s times, and we can still write

$$a_x^n = (a_1x_1+a_2x_2)^n,$$

provided we remember that we have made no stipulation as to the way the a's combine in multiplication. Most of what precedes is, of course, well known, but it was necessary to show how the symbolic notation of Clebsch and Aronhold could be applied to unsymmetrical forms. We have now to see how a linear form can itself be written as a product. I call to mind that, if we use Grassmann's methods, we replace a set of variables $x_1, x_2 \dots x_n$ by a single complex variable

$$x = x_1e_1+x_2e_2+\dots+x_ne_n,$$

where $e_1 \dots e_n$ are "units" supposed to obey the polar law of multiplication, so that we have

$$e_ie_j = -e\,e_i,$$

$$e_i^2 = 0,*$$

and then we have for any sets

$$xy = -yx,$$

$$x^2 = 0.$$

The product of all the units is a scalar, and is assumed to be unity.

* It should be noticed that the second part of this polar law is not a consequence of the first, and that consequently the late Prof. Smith's objection to Clifford's system ($e_i^2 = -1$, $e_ie_j = -e_je_i$) does not seem to be valid. The same objection would apply to quaternions.

If we take any unit e_1, the product of the remaining units is the *conjugate* of e_1, and is denoted by Ke_1. We have

$$e_1 Ke_1 = 1,$$

and this is taken as the definition of K, viz., we have

$$e_i \,.\, Ke_i = 1,$$

$$e_j \,.\, Ke_i = 0.$$

Ke_i is obviously the product of the remaining units $e_1 \ldots e_{i-1},\ e_{i+1} \ldots e_n$ taken in such an order as to satisfy the first of the above equations. In this paper I use instead of Ke_i another quantity ϵ_i defined by the equation

$$\epsilon_i e_i = 1,$$

$$\epsilon_i e_j = 0\,;$$

we have, obviously, $\epsilon_i = (-)^{n-1} Ke_i.$

And $\epsilon_i(x_1 e_1 + x_2 e_2 + \ldots + x_n e_n) = x_i.$

Now consider the linear form

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n.$$

Let $x = x_1 e_1 + \ldots + x_n e_n,$

then $x_i = \epsilon_i x,$

and
$$\begin{aligned} a_1 x_1 + a_2 x_2 + \ldots + a_n x_n &= a_1 \epsilon_1 x + a_2 \epsilon_2 x + \ldots + a_n \epsilon_n x \\ &= (a_1 \epsilon_1 + a_2 \epsilon_2 + \ldots + a_n \epsilon_n)\, x, \end{aligned}$$

and we see that any linear form can be written as the product of two factors, one factor containing the variables, and the other containing the coefficients. Now a lineo-linear form was written above as the product of two linear forms, and we see now that it can be written as the product of four factors, two involving the coefficients, and two involving the variables. It is therefore necessary to see how the factors combine in multiplication.

Let the linear forms be

$$a_x = (a_1 \epsilon_1 + a_2 \epsilon_2)(x_1 e_1 + x_2 e_2),$$

$$a_y = (a_1 \epsilon_1' + a_2 \epsilon_2')(y_1 e_1' + y_2 e_2').$$

Now, we know that $\epsilon_1 \epsilon_2 = -\,\epsilon_2 \epsilon_1,$

and, as regards $\epsilon_1 \epsilon_1',\ \epsilon_2 \epsilon_2',\ \epsilon_2 \epsilon_1',\ \epsilon_1 \epsilon_2'$, we stipulate that these products

shall follow the commutative law; so that we have the following convention: *If we have any number of pairs* ($\epsilon_1, \epsilon_2;\ \epsilon_1', \epsilon_2';\ \ldots$) *the elements of each pair combine according to the polar law, but combine with the elements of every other pair according to the commutative law.**

We have found that we can write

$$a_x = (a_1\epsilon_1 + a_2\epsilon_2)\,x.$$

Now, in all that follows, we shall only have to consider the coefficients of the forms we have to deal with, and we can therefore confine our attention to the first of the two factors giving a_x, that is to say, instead of working with the linear form $a_1x_1 + a_2x_2$, we can work with the form $a_1\epsilon_1 + a_2\epsilon_2$; or, in other words, we can consider all linear forms, and therefore forms of any order, as involving *polar* variables, instead of *scalars*.

I shall now change the notation, and shall use any letters to denote the two polar variables in a binary linear form; thus, for instance, I write as before,

$$a_u = a_1u_1 + a_2u_2;$$

but it must be remembered that u_1, u_2 are, not scalars, but *polars*,† and that we must multiply a_u by another polar, if we are to get a linear form involving scalars.

II. *The Fundamental Theorem.*

Now, suppose we take two linear forms

$$a_u = a_1u_1 + a_2u_2,$$

$$b_u = b_1u_1 + b_2u_2,$$

and multiply them together, we get, since

$$u_1^2 = u_2^2 = 0, \quad u_1u_2 = -u_2u_1 = 1,$$

$$a_ub_u = a_1b_2 - a_2b_1 = (ab).$$

* This is not Clifford's convention; he makes the elements of different pairs combine according to the polar law; but, if we do this, we get into endless difficulties with the signs, and, as a matter of fact, several of Clifford's signs are wrong; with the convention in the text the signs of all forms can be determined without difficulty.

† There seems no obvious reason why *polar* should not be used as a noun, and it would simplify matters considerably.

Now, suppose we take two quadric* forms

$$a_{uv} = a_{11}u_1v_1 + a_{12}u_1v_2 + a_{21}u_2v_1 + a_{22}u_2v_2,$$

$$b_{uv} = b_{11}u_1v_1 + b_{12}u_1v_2 + b_{21}u_2v_1 + b_{22}u_2v_2.$$

If we write $$a_u = a_1u_1 + a_2u_2,$$

we have $$a_{uv} = a_u a_v,$$

$$b_{uv} = b_u b_v,$$

and, therefore,

$$a_{uv}b_{uv} = a_u a_v b_u b_v = a_u b_u \,.\, a_v b_v = (ab)(ab) = (ab)^2,$$

where we must remember that

$$(a_1b_2 - a_2b_1)^2 = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11}.$$

In the same way we should get

$$a_{uvw}b_{uvw} = (ab)^3.$$

And obviously $$a_u a_z \,.\, b_u b_z = (ab)\, a_z b_x,$$

$$a_u a_v a_w \,.\, b_u b_x b_y = (ab)\, a_v a_w \,.\, b_x b_y,$$

and $$a_u a_v a_w \,.\, b_u b_v b_y = (ab)^2\, a_w b_y.$$

In working with unsymmetrical forms, we must be careful to keep the variables in their right places, since $a_u a_v$ and $a_v a_u$ are not identical; thus

$$a_z a_u \,.\, b_x b_u = a_z b_x\,(a_u b_u) = a_z b_x\,(ab),$$

$$a_z a_u \,.\, b_u b_x = a_z\,(ab)\, b_x.$$

We have found that

$$a_u a_v a_w a_{w'} \ldots\, b_u b_v b_z b_{z'} \ldots = (ab)^2\, a_w a_{w'} \ldots\, b_z b_{z'} \ldots.$$

Now, if the forms a, b are symmetrical, the right-hand side is obviously the $2\,(n-2)$-tuply linear form answering to $(ab)^2 a_x^{n-2} b_x^{n-2}$; that is to say, if we multiply together two multiply linear forms having two pairs $(u_1, u_2;\ v_1, v_2)$ of polars in common, we get the second alliance (*Ueberschiebung*) of the quantics answering to the forms. And in the same way we see, generally, that *if we multiply*

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* I follow Clifford in classifying forms according to the order of the ultimate form obtained by introducing scalars, and making the set of variables identical: thus, an n-tuply linear form is an n-thic form.

together two multiply-linear forms, having r pairs of polars in common, we get the r^{th} *alliance of the quantics answering to the forms.**

III. *Graphs.*

We have now to consider the graphical representation of forms; suppose we have an n-tuply linear form, this is represented in the same way as an n-valent atom in chemistry; viz., by a small circle with n rays or bonds proceeding from it, each ray answering to a pair of polars; if we multiply together two forms having r pairs in common, we connect their representative atoms by r bonds; thus, if we take two cubics $a_x a_y a_z$, $b_x b_y b_w$, we represent their second alliance $(ab)^2 a_z b_w$ in the way shown in Fig. (1), where one atom is black to distinguish it from the other.

It is clear, without any formal proof, that what precedes can be extended to any number of forms, so that, if we take any concomitant written in its symbolic form, we can write down the corresponding graph. Thus, the discriminant of a cubic is $(ab)^2 (cd)^2 (ac) (bd)$; it is the result of the following multiplication,†

$$a_x a_y a_z \,.\, b_x b_y b_w \,.\, c_u c_v c_z \,.\, d_u d_v d_w\,,$$

and we get the graph in Fig. (2), and it is obvious that we could have got it by putting down four "atoms" answering to the four forms a, b, c, d, and joining two atoms (a, b) by a bond for every time (ab) occurs as a factor. In the same way we see that $(ab)^2 (bc)^2 (ca)^2$, the cubic invariant of a quartic, is represented by Fig. (3), and that $(ab)^2 (bc)\, a_x c_x^2$ is represented by Fig. (4).

It must not be forgotten that, in the first instance, a graph does not represent the quantic, but a certain polarised blank form of the quantic; thus, in Fig. (5), the graph does not represent the cubic a_x^3, but the product $a_u a_v a_w$, which gives, in the first instance, $a_x a_y a_z$, and then a_x^3 when we make the three pairs of variables identical.

IV. *Links and their Properties.*

The determinant $x_1 y_2 - x_2 y_1$ will be denoted by (xy) and represented by the graph Fig. (6); such determinants will be called *links*.

If we square (xy), we get

$$(x_1 y_2 - x_2 y_1)^2 = x_1^2 y_2^2 - x_1 x_2 y_2 y_1 - x_2 x_1 y_1 y_2 + x_2^2 y_1^2 = 2 x_1 x_2 y_1 y_2 = 2.$$

* "*Quære*, whether this beautiful use of the method of polar multiplication is not, in its ultimate essence, identical with Professor Cayley's original method of hyperdeterminants."—Prof. Sylvester, *American Journal*, I., 128.

† In what follows, the original quantic we work with will always be supposed symmetrical.

We also find

$$(xy)(zy) = (x_1y_2 - x_2y_1)(z_1y_2 - z_2y_1) = x_1z_2 - x_2z_1 = (xz),$$

and therefore $$(yx)\,(yz) = (xz).$$

If we multiply a_x by (yx), we get

$$(a_1x_1 + a_2x_2)(y_1x_2 - y_2x_1) = a_1y_1 + a_2y_2,$$

or $$a_x\,(yx) = a_y.$$

We have $$a_xb_y = a_1b_1x_1y_1 + a_1b_2x_1y_2 + a_2b_1x_2y_1 + a_2b_2x_2y_2.$$

Therefore $$a_xb_y\,(xy) = a_1b_2 - a_2b_1 = (ab) = a_xb_x,$$

and therefore in any product of this kind, when we multiply by (xy), we need only change a factor b_y into b_x.*

V. *Quadrics, Skew and Symmetrical.*

Consider the quadric

$$a_xa_y = a_{11}x_2y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2;$$

if we multiply this by $x_1y_2 - x_2y_1$, we get

$$a_{12} - a_{21},$$

and therefore $$a_xa_y\,(xy)$$

vanishes if, and only if, a_xa_y is symmetrical, and, since there is nothing to prevent the coefficients a_{ik} from involving polars, we see that any form a containing x, y is symmetrical with respect to these two variables, if, and only if, $a\,(xy)$ vanishes. Now, if a form is to be symmetrical with respect to all the variables involved, it is obviously necessary and sufficient that it should be symmetrical with respect to every pair of variables; that is to say, *the necessary and sufficient condition that a form should be symmetrical is that if we multiply the form by all the links formed by pairing its variables, each of the products must vanish.*

We have found the condition that a quadric may be symmetrical, that is, that we may have

$$a_xa_y = a_ya_x.$$

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* It would not have been enough to say that $a_xb_y\,(xy) = a_x.\,\overline{b_y}\,(xy) = a_xb_x = (ab)$, since we stipulated that x, y combined according to the commutative law.

We have now to find the condition that it may be skew, that is, that we may have
$$a_x a_y = -a_y a_x.$$
This gives
$$a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2 = -a_{11}x_1y_1 - a_{12}x_2y_1 - a_{21}x_1y_2 - a_{22}x_2y_2,$$
and therefore
$$a_{11} = a_{22} = 0,$$
$$a_{12} = -a_{21},$$
and the quadric reduces to
$$a_{12}(x_1y_2 - x_2y_1);$$
that is to say, if a quadric is skew, it is a multiple of the link of its variables; and, in the same way as before, we see that, if any form is skew as regards any pair of variables, it is a multiple of the link of this pair of variables.

It is always easy to determine the coefficient of the link; for, if we have
$$f(x, y \ldots) = A\,(xy),$$
we get, by multiplying by (xy),
$$f(x, x \ldots) = 2A,$$
and therefore
$$f(x, y \ldots) = \tfrac{1}{2} f(x, x \ldots)\,(xy).$$
Thus
$$a_z a_x \,.\, b_z b_y$$
is a skew function, if a, b refer to the same quadric, for, if we interchange x and y, we get
$$a_z a_y \,.\, b_z b_x.$$
Now, if we interchange a and b, the original form becomes
$$b_z b_x \,.\, a_z a_y,$$
and
$$b_x a_y = a_y b_x,$$
$$b_z a_z = -a_z b_z,$$
and therefore
$$a_z a_y \,.\, b_z b_x = -b_z b_x \,.\, a_z a_y = -a_z a_x \,.\, b_z b_y.$$
Therefore $a_z a_y \,.\, b_z b_x$ is a skew form, and the coefficient of (xy) is
$$\tfrac{1}{2} a_z a_y \,.\, b_z b_y = \tfrac{1}{2}(ab)^2.$$
In the same way, we see that the form in Fig. (7) is skew, and that twice the coefficient of (xy) is the graph in Fig. (8), that is to say, the quadric invariant (i) of the quartic.

VI. *Unsymmetrical Forms.*

Consider the graph Fig. (9). We see at once that its symbolical form is $(ab)^2 a_x^2 b_x^2$, and that it answers to the Hessian of a quartic; but, before we can identify the two, we must see whether the graph is symmetrical. Now, if we start with a symmetrical quartic, the graph is obviously unchanged if we write x for y, or u for v; to see whether it is unaltered when we interchange x and u, we must multiply $(ab)^2 a_x a_y . b_u b_v$ by (xu). Now we know that this comes to the same thing as identifying x and u, so that the product is

$$(ab)^3 a_y b_v,$$

and the graph for this is Fig. (7), and we know that this is

$$\frac{i}{2}(yv),$$

and does not vanish; we see, then, that Fig. (9) is not symmetrical with respect to x and u, and in the same way we see that it is not symmetrical with respect to y and v, or x and v, or y and u. Now, there is an essential and obvious difference between symmetrical and unsymmetrical graphs. Suppose we have two symmetrical quartic graphs, one having $xyzw$ as its variables, and the other having $stuv$; if we form the second alliance of these graphs, it is obviously a matter of indifference which pair of letters in the one we identify with a pair of letters in the other; if, however, the graphs are not both symmetrical, this is not the case, and we get different results according to the way we combine them. Thus, a sextic has an unsymmetrical quartic covariant represented by Fig. (10), where the broad bond denotes four bonds; if we join this to the sextic by the bonds u, v,* we get Fig. (11); if we join it by (x, u), we get Fig. (12). Now, these two covariants are not identical, for it can be shown that

$$(11) = (12) + \frac{A}{2} f,$$

where f is the sextic, and A is the invariant $(ab)^6$.

This distinction between symmetrical and unsymmetrical forms is of the greatest importance. If we work with unsymmetrical forms, we have to be extremely careful not to identify results got by combining them in different ways; four pages of Clifford's "Fragments" relate

* That is to say, if we multiply it by $a_u a_v \ldots$; I shall use this abbreviated phrase throughout the rest of this paper.

to the form $\kappa f+\lambda g$, where f is a quintic, and g its covariant of degorder (3, 5); now, all the results obtained on these pages are vitiated by the fact that Clifford has treated unsymmetrical forms as if they were symmetrical. We must remember that every form is to be supposed unsymmetrical until it is proved to be symmetrical.

I proceed to show how we can always get a symmetrical form answering to a given unsymmetrical form.

VII. *Reduction of Graphs to Symmetrical Forms.*

Suppose we have any unsymmetrical form, and that we wish to make it symmetrical; the symmetrical form will obviously be obtained by taking the arithmetical mean of all the different values of the form. Thus, in the case of the quartic, we get an unsymmetrical quartic covariant $h = (ab)^2 a_x a_y . b_u b_v$: it is easy to see that, if a is a symmetrical form, we get six different values for h; for h is obviously unchanged if we interchange x, y, or u, v; so that the twenty-four possible values obtained by permuting xy, uv reduce to six. Let H be the symmetrical form of h, then we have

$$\begin{aligned} 6H = {} & (ab)^2 a_x a_y b_u b_v + (ab)^2 a_u a_y b_x b_v + (ab)^2 a_x a_v b_u b_y \\ & + (ab)^2 a_v a_y b_x b_u + (ab)^2 a_x a_u b_y b_v + (ab)^2 a_u a_v b_x b_y \\ = {} & 6\,(ab)^2 a_x a_y b_u b_v + \Sigma \left\{ (ab)^2 a_u a_y b_x b_v - (ab)^2 a_x a_y b_u b_v \right\}, \end{aligned}$$

where the sign Σ denotes the sum of the terms obtained by subtracting the first term of $6H$ from each of the others.

$$\begin{aligned} \text{Now, } (ab)^2 a_u a_y b_x b_v - (ab)^2 a_x a_y b_u b_v &= (ab)^2 a_y b_v (a_u b_x - a_x b_u) \\ &= (ab)^3 a_y b_v (ux) \\ &= \tfrac{1}{2} (ab)^3 (ux)(a_y b_v - a_v b_y), \end{aligned}$$

if we interchange a, b, and take the semi-sum of the two expressions;

and this is $\qquad \frac{1}{2}(ab)^4 (ux)(yv)$.

In the same way, we get $\frac{1}{2}(ab)^4(ux)(yv)$ from the next term, and $\frac{1}{2}(ab)^4(xv)(uy)$ from each of the next two; the last difference

$$a_x a_y b_u b_v - a_u a_v b_x b_y$$

vanishes, since the second term reduces to the first if we interchange a and b.

We have, therefore,

$$6H = 6\,(ab)^2 a_x a_y b_u b_v - (ab)^4 (xu)(yv) - (ab)^4 (xv)(yu),$$

or $$H = (ab)^2 a_x a_y b_u b_v - \frac{(ab)^4}{6} \{(xu)(yv)+(xv)(yu)\}.*$$

In the same way, the quintic has an unsymmetrical quintic covariant

$$g = (ab)^4 (bc)\, a_x c_t c_u c_v c_w.$$

If c is symmetrical, the different values of g are got by interchanging x with t, u, v, w. We therefore have, if G is the symmetrical form of g,

$$\begin{aligned} 5G &= (ab)^4 (bc) \{a_x c_t c_u c_v c_w + a_t c_x c_u c_v c_w + a_u c_t c_x c_v c_w \\ &\qquad + a_v c_t c_u c_x c_w + a_w c_t c_u c_v c_x\} \\ &= 5\,(ab)^4 (bc)\, a_x c_t c_u c_v c_w + \Sigma (ab)^4 (bc) \{a_t c_x c_u c_v c_w - a_x c_t c_u c_v c_w\}, \end{aligned}$$

where, as before, Σ denotes the sum of the four terms obtained by subtracting the first term of $5G$ from each of the others.

Now, $$\begin{aligned} &(ab)^4 (bc)(a_t c_x c_u c_v c_w - a_x c_t c_u c_v c_w) \\ &= (ab)^4 (bc)\, c_u c_v c_w (a_t c_x - a_x c_t) \\ &= (ab)^4 (bc)(ac)\, c_u c_v c_w (tx). \end{aligned}$$

And therefore

$$\begin{aligned} G &= (ab)^4 (bc)\, a_x c_t c_u c_v c_w - \tfrac{1}{5}\Sigma (ab)^4 (bc)(ac)\, c_u c_v c_w (xt), \\ &= g(x, t, u, v, w) - \tfrac{1}{5}\Sigma j(u, v, w)(xt), \end{aligned}$$

where $j(u, v, w)$ denotes the covariant $(5, 3;\ 3)$,

$$(ab)^4 (bc)(ac)\, c_u c_v c_w.$$

As a last example, I take the covariant $\vartheta\,(2, 1;\ 3, 1;\ 3)$ of a cubic and quadric. If the cubic is a, and the quadric is α, we have

$$\vartheta\,(x, yz) = (\alpha a)\, \alpha_x a_y a_z.$$

Now, here the only changes which can affect the form of ϑ are the interchange, firstly of x and y, and secondly of x and z; and therefore we shall have, if Θ is the symmetrical form of ϑ,

$$3\Theta = 3\vartheta\,(x, yz) + \{\vartheta\,(y, xz) - \vartheta\,(x, yz)\} + \{\vartheta\,(z, xy) - \vartheta\,(x, yz)\}.$$

Now $$\vartheta\,(y, xz) - \vartheta\,(x, yz)$$

is a skew function of x, y, and therefore divides by (xy). We have

$$(\alpha a)\, \alpha_y a_x a_z - (\alpha a)\, \alpha_x a_y a_z = A\,(xy).$$

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* Clifford, p. iii.

Multiplying by (xy), we get

$$-2\,(\alpha a)^2\,a_z = 2A,$$

or

$$A = -\,(\alpha a)^2\,a_z.$$

We therefore have

$$\Theta = (\alpha a)\,\alpha_x\alpha_y a_z - \tfrac{1}{3}\Sigma\,(\alpha a)^2\,a_z\,(xy).$$

This result might, of course, have been found in the same way as the other two; I have adopted another method, partly for the sake of variety, and partly because this last method is generally more convenient when we are working with graphs.*

VIII. *Discriminations.*

It is, in general, easy to see when a graph vanishes, or is skew or symmetrical with respect to any pair of letters. Thus, consider the graph in Fig. (13); we can see at once that this vanishes, for it represents the covariant (5, 3; 1),†

$$(ab)^3\,(ac)^2\,(bc)^2\,c_x.$$

This covariant changes its sign if we interchange a and b, and therefore vanishes. In the same way, if we consider Fig. (14), we see that this is symmetrical with respect to (x, y); for it is

$$(ab)^2\,(ac)^2\,(bc)^2\,a_x b_y c_z,$$

and the interchange of x and y is identical with the interchange of a and b, and therefore leaves the graph unchanged. On the other hand, the graph in Fig. (15) is skew in (x, y); for it is

$$(ab)^3\,(ac)(bc)\,a_x b_y c_u c_v c_w,$$

and the interchange of x and y gives

$$(ab)^3\,(ac)(bc)\,a_y b_x c_u c_v c_w,$$

which is what the original covariant becomes when we interchange a and b and change the sign.

Clifford says (p. iii.) that the graph in Fig. (16) is a multiple of (xy); but we can see at once that it is not skew, but symmetrical, since it is

$$(ab)^2\,(ac)(bc)\,a_x b_y c_u c_v,$$

* The symmetrical forms of g, ϑ, on Clifford's pp. ix., iii., are not correct.

† Covariant of quintic of degorder (3, 1).

and the interchange of x and y leaves it unaltered. From this Clifford infers that it vanishes (which is not the case),* and this leads to some other incorrect results.

For triangular graphs we get the following rules :—

1. If two saturated† vertices of a triangle are joined by an odd number of bonds, and are joined to the third vertex by any the same number of bonds, the graph vanishes.

2. If two vertices, each having one free bond, are joined by an odd number of bonds, and are joined to the third vertex by any the same number of bonds, the graph is a multiple of the link of the free bonds at the two first mentioned vertices, the coefficient being one-half the graph obtained by joining these two vertices by an additional bond.

3. If two vertices, each having one free bond, are joined by an even number of bonds, and are joined to the third vertex by any the same number of bonds, the graph is symmetrical with respect to the free bonds at the two first mentioned vertices.

As a particular case of (1), we see that, if two saturated atoms are connected by an odd number of bonds, the resulting invariant vanishes.

If two atoms, having one free bond each, are connected by an odd number of bonds, the resulting quadric covariant is skew.

If two atoms, having one free bond each, are connected by an even number of bonds, the resulting quadric covariant is symmetrical.

IX. *Elementary Reductions.*

Suppose we have any form $f(xu)$ containing two letters, and as many besides as we please; then we have

$$f(x, u) = \tfrac{1}{2}\{f(x, u)+f(u, x)\} + \tfrac{1}{2}\{f(x, u)-f(u, x)\}.$$

Now, the first bracket is obviously a symmetrical function of x, u, and the second bracket is a skew function, and we see that every form containing two letters can be decomposed into two parts, one symmetrical and the other skew with respect to these two letters, and this decomposition is unique.‡

* If the graph were a multiple of (xy), the coefficient would be $\frac{1}{2}(ab)^3(ac)(bc)c_uc_v$, which vanishes, and therefore the graph would vanish.

† In Fig. (13), the vertices at the base of the triangle are saturated, and the third angle has one free bond; in Fig. (16), the two vertices at the base have respectively the free bonds x, y.

‡ Let a be any quantity, E any distributive operator, then a can be decomposed uniquely into two parts, α, β, such that $E\alpha=\lambda\alpha$, $E\beta=\mu\beta$, λ, μ being unequal scalars, for we have $a=\alpha+\beta$, $Ea=\lambda\alpha+\mu\beta$, and α, β are determined uniquely.

In what follows $(A : xu, yv)$ denotes a function which is unaltered when we interchange x, u, and also when we interchange yv; $(A : xu)$ means a function which is unaltered when we interchange x, u*; $A\,(xu)$ denotes, as before, a multiple of the link (xu).

We have seen that we can write

$$f(x, y, u, v) = A\,(xu) + (A' : xu),$$

where, of course, A, A' involve yv; transforming these in the same way, we get

$$A = B_1\,(yv) + (B_2 : yv),$$

$$A' = B_1'\,(yv) + (B_2' : yv),$$

and therefore

$$f(x, y, u, v) = B_1\,(xu)(yv) + (B_2 : yv)(xu) + (B_3 : xu)(yv) + (B_4 : xu, yv).$$

If f contains more than four letters, we can, of course, go on in this way, and it is easy to see the form of the general result. The formula just given is enough for the purposes of the present paper.

Now, suppose that f is known to change sign when we interchange x, u, and also interchange y, v; we have

$$f = B_1\,(xu)(yv) + (B_2 : yv)(xu) + (B_3 : xu)(yv) + (B_4 : xu, yv),$$

$$-f = B_1\,(ux)(vy) + (B_2 : vy)(ux) + (B_3 : ux)(vy) + (B_4 : ux, vy),$$

and, remembering the meanings of $(B_2 : yv)$, &c., and that (xu), (yv) are skew, we get

$$0 = B_1\,(xu)\,(yv) + (B_4 : xu, yv),$$

and

$$f = (B_2 : yv)\,(xu) + (B_3 : xu)\,(yv) \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

If f is unaltered when we interchange x, u, and also y, v, we shall find in the same way

$$f = B_1\,(xu)(yv) + (B_4 : xu, yv) \ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

Multiplying (α) by (yv), we get, if f was $f(x, y, u, v)$,

$$f(x, y, u, y) = 2\,(B_3 : xu).$$

Multiplying it by (xu), we get

$$f(x, y, x, v) = 2\,(B_2 : xu).$$

* This is not a good notation, but I have been unable to devise another that should look better, and at the same time guard against all risk of confusion.

If f is $(f : xy, uv)$, B_2, B_3 are obviously the same forms. Thus, consider the form $$(ab)\, a_x a_y\, b_u b_v.$$

The substitution $(xu)(yv)$* is obviously equivalent to the interchange (ab), together with a change of sign, and we can therefore use (α).

This gives $$(ab)\, a_x a_y b_u b_v = B_2\,(xu) + B_3\,(yv).$$

To determine B_2 multiply by (yv), and we get

$$(ab)^2\, a_y b_v = 2B_2,$$

and in the same way $$(ab)^2\, a_x b_u = 2B_3;$$

and we have therefore

$$(ab)\, a_x a_y b_u b_v = \tfrac{1}{2}\,(ab)^2\, a_x b_u\,(yv) + \tfrac{1}{2}\,(ab)^2\, a_y b_v\,(xu).$$

This equation is represented graphically in Fig. (17).

X. *Form-Systems.*

We have now all the materials we require for the graphic construction of the form-system of a quantic.

I call to mind that forms are classified according to their degree and weight. If we write down the graph corresponding to a given form, the degree of the form is the number of atoms in the graph, and its weight is the total number of bonds connecting the atoms; thus Fig. (16) represents a form of order four, degree three, and weight four, appertaining to a quartic.†

Moreover, if we consider the reductions of graphs already given, we see that, if a graph of weight w reduces, as in Fig. (17), to a sum of links, the coefficients of the links are at least of weight $w+1$, and that, if we find the symmetric form answering to a given graph, of weight w, the two forms differ by a sum of products of links and forms of weight $w+1$, at least.

Now, the way we construct a form-system is as follows: Suppose we start with an n-thic. Joining this to itself by n bonds, we get the heaviest‡ form of the second degree; joining the quantic to itself by $n-1$, $n-2$ ... bonds, we get all the forms of the second degree

* I use here the ordinary notation for cyclic substitutions; the word *interchange* or *substitution* will always be used when this is the case, to prevent confusion.

† This rule is easily seen to be correct by considering the symbolic expression answering to the graph.

‡ It seems natural, and is certainly convenient, to describe a graph (or form) as *heavier* than a form of less weight.

arranged according to their weights, in descending order. Some of these forms (all the forms of odd weight) will reduce to sums of links, and these reductions must be effected, and unsymmetrical graphs must be made symmetrical. After finding in this way all the forms of the second degree, we get the forms of the third degree by joining the n-thic to these forms of the second degree, beginning with the heaviest, and in each case we form the combinations in the order of their weights, beginning with the heaviest.

We have now to see what happens when we join the quantic (f) by r bonds to a form reducible to a sum of links, or to an unsymmetric form. Suppose we have a form ϕ of any degree, and of weight w, and that this contains a term

$$\psi\,.\,(xu),^*$$

where ψ is of weight $w+1$; then ϕ gives a form of weight $w+r$, and, as regards the term just mentioned, three cases may present themselves:—

1. x, u may both be among the r bonds by which we join f to ϕ; then, since f is supposed symmetrical,

$$(xu)\,f\,(x,\,u,\,\ldots)=0,$$

and the term contributes nothing.

2. Let one of the two letters, say x, be among the r bonds; the factor (xu) changes x to u, and we have to join f to ψ, by $r-1$ bonds, and the weight of the resulting term is

$$w+1+r-1=w+r.$$

3. Let neither x nor u be among the r bonds; then we have to join x to ψ by r bonds, and we get the product of (xu) and a form of weight $w+r+1$.

In the third case, the form of weight $w+r+1$ will have been obtained before we got down to the forms of weight $w+r$, and need not be considered.

In the second case, the form of weight $w+r$ is got by joining f to a form of weight $w+1$, and all the combinations of this form with f will have been disposed of before we got down to forms of weight w.

We see, then, that if we arrange our forms in the order agreed

* It must be remembered that, since (Clebsch, p. 8) the coefficients and the variables are transformed by inverse substitutions, the weight of a link may be taken to be -1.

upon, we may neglect all forms reducing to sums of links, since everything that such forms could furnish will have been obtained before we get to them, and that we can neglect the complementary part* of any unsymmetric graph; or, in other words, that we can treat any graph as symmetrical, and join f to it in whatever way may happen to be most convenient.

It must be noticed that we are not at liberty to reject a graph because it contains, as part of itself, a graph reducible to a sum of links; this is simply because the reducible part is lighter than the graph from which the whole graph was derived. If, however, a graph contains, as part of itself, a graph reducing to a sum of products of links and invariants, it may obviously be rejected.

I proceed to apply these principles to the theory of quadric, cubic, and quartic forms.

SEC. 1. *Quadric.*

The forms of the second degree are given in Figs. (18, 19); the reduction in Fig. (19) is obvious, and we see that there can be no irreducible forms of the third degree.

SEC. 2. *Cubic.*

The forms of the second degree are given in Figs. (20—22); of these (20) vanishes, (21) is symmetrical [for, if we multiply it by (xy), we get (20)], (22) reduces by Fig. (17). We need, therefore, only consider (21), which is the Hessian.†

The forms of the third degree obtained from (21) are given in Figs. (23, 24); of these (23) vanishes, as I proceed to show. Written symbolically, the graph is

$$(ab)^2\,(bc)\,(ac)\,c_x;$$

we get three forms of this by interchange of a, b, c, and, taking one-third of the sum of these, we get

$$\begin{aligned}(ab)^2\,(bc)(ac)\,c_x &= \tfrac{1}{3}\left\{(ab)^2\,(bc)(ac)\,c_x+(bc)^2\,(ca)(ba)\,a_x\right.\\ &\qquad\left.+(ca)^2\,(ab)(cb)\,b_x\right\}\\ &= -\frac{(bc)(ca)(ab)}{3}\left\{(ab)\,c_x+(bc)\,a_x+(ca)\,b_x\right\}=0.\ddagger\end{aligned}$$

* The complementary part of an unsymmetric graph is the sum of links that has to be added to make it symmetrical.

† Δ in Clebsch's notation; I use Clebsch's notation throughout for all the forms considered.

‡ I have given the above proof instead of Clifford's; Clifford proves that (23) vanishes, by reasoning of which I am unable to see the force.

Now that we have proved that (23) vanishes, we can see that (24) is symmetrical, for, if we multiply it by (xu), we get Fig. (23); (24) is therefore a symmetrical covariant of degorder (3, 3), and is therefore what Clebsch denotes by Q. The covariants of the fourth degree derived from Q are given in Figs. (25—27). Of these (25) is an invariant, the discriminant (R). The reduction in Fig. (26) is obvious; (27) is got by joining the Hessian to Fig. (22), and is found at once to be reducible, in the way shown in the figure; we see that the only irreducible form of the fourth degree is an invariant, and that there are, therefore, no irreducible forms of higher degrees, so that the form system of the cubic consists of the forms f, Δ, Q, R.

Sec. 3. *Quartic.*

The forms of the second degree are given in Figs. (28—31). (28) is the invariant i; the reduction in (29) is obvious; (30) is unsymmetrical, and its symmetrical form has already been found to be (32); this symmetrical form is the Hessian (H); (31) reduces in the way shown in the figure. I shall go through the calculation here, as Fig. (31) does not agree with Clifford's results. The symbolic form of (31) is

$$(ab)\, a_x a_y a_z b_u b_v b_w .$$

This form changes sign if we effect the substitution $(xu)(yv)(zw)$; and therefore, when we expand it in a series of links, we need only keep the skew terms; we have, therefore,

$$(ab)\, a_x a_y a_z b_u b_v b_w = A\,(xu)(yv)(zw) + \Sigma\,(B : yv,\, zw)(xu).$$

Multiplying by $(xu)(yv)(zw)$, we get

$$(ab)^4 = 8A.$$

Multiplying by (xu), we get

$$(ab)^2\, a_y a_z b_v b_w = 2A\,(yv)(zw) + 2\,(B : yv,\, zw),$$

and therefore

$$(B : yv,\, zw) = \tfrac{1}{2}\,(ab)^2\, a_y a_z b_v b_w - A\,(yv)(zw),$$

and we get similar values for the other B's; substituting and reducing, we get

$$(ab)\, a_x a_y a_z b_u b_v b_w = \tfrac{1}{2}\,\Sigma\,(ab)^2\, a_y a_z b_v b_w\,(xu) - \frac{i}{4}\,(xu)(yv)(zw).*$$

* Clifford only gets the first part of this; but it can easily be verified directly that the above result is correct.

The forms of the third degree got from H are given in Figs. (33—36). Of these (33) is the invariant j; the reduction in Fig. (34) is obvious. I shall follow Clifford in showing that (35) is reducible, but I shall not use graphs in reducing it, as it is easier to get the signs right if we use symbolic methods. We have

$$(ab)\,a_x a_y a_z\, b_u b_v b_w = \tfrac{1}{2}\,(ab)^2\, a_y a_z\, b_v b_w\,(xu) + \tfrac{1}{2}\,(ab)^2\, a_x a_z\, b_u b_w\,(yv)$$
$$+\tfrac{1}{2}\,(ab)^2\, a_x a_y\, b_u b_v\,(zw) - \frac{i}{4}\,(xu)(yv)(zw).$$

Now, multiply this *into* $c_x c_y c_u c_t$; we get

$$(ab)(ac)^2\,(bc)\,a_z b_v b_w c_t = \tfrac{1}{2}\,(ab)^2\,(ac)(bc)\,a_z b_w c_v c_t$$
$$+\tfrac{1}{2}\,(ab)^2\,(ac)^2\,(bc)\,b_v c_t\,(zw)$$
$$= -\tfrac{1}{2}\,(ac)^2\,(ab)(bc)\,a_z b_v b_t c_w$$
$$+\tfrac{1}{2}\,(ab)^2\,(ac)^2\,(bc)\,b_v c_t\,(zw).$$

Now $\quad (ab)(ac)^2\,(bc)\,a_z b_v b_w c_t - (ab)(ac)(bc)\,a_z b_v b_t c_w$

$$= (ab)(ac)^2\,(bc)\,a_z b_v\,(b_w c_t - b_t c_w)$$
$$= (ab)(ac)^2\,(bc)^2\,a_z b_v\,(wt),$$

and therefore, if we compare our results, we get

$$-\tfrac{1}{2}\,(ac)^2\,(ab)(bc)\,a_z b_v b_t c_w + \tfrac{1}{2}\,(ab)^2\,(ac)^2\,(bc)\,b_v c_t\,(zw)$$
$$= (ac)^2\,(ab)(bc)\,a_z b_v b_t c_w + (ac)^2\,(bc)^2\,(ab)\,a_z b_v\,(wt);$$

and therefore

$$-\tfrac{3}{2}\,(ab)(ac)^2\,(bc)\,a_z b_v b_t c_w$$
$$= \tfrac{1}{2}\,(ab)^2\,(ac)^2\,(bc)\,b_v c_t\,(zw) - (ab)(ac)^2\,(bc)^2\,a_z b_v\,(wt).\text{*}$$

But $\quad (ab)^2\,(ac)^2\,(bc)\,b_v c_t = \tfrac{1}{2}\,(ab)^2\,(ac)^2\,(bc)^2\,(vt),$

and therefore $\quad = \tfrac{1}{2}j\,(vt),$

$$-\tfrac{3}{2}\,(ab)(ac)^2\,(bc)\,a_z b_v b_t c_w = \frac{j}{2}\left\{(vt)(zw) - \frac{(zv)(wt)}{2}\right\}$$
$$= \frac{j}{2}\left\{(vt)(zw) + \frac{(vw)(zt)-(zw)(vt)}{2}\right\}$$

---

* Clifford finds

$$\tfrac{1}{2}\,(ab)(ac)^2\,(bc)\,a_z b_v b_t c_w = (ab)(ac)^2(bc)^2\,a_z b_v\,(tw) - \tfrac{1}{2}\,(ab)(ac)^2\,(bc)^2\,b_v c_t\,(zw).$$

[since $(zv)(wt)+(vw)(zt)+(wz)(vt)=0$]

$$=\frac{j}{4}\left\{(vt)(zw)+(zt)(vw)\right\},$$

or
$$(ab)(ac)^2(bc)\,a_zb_vb_tc_w=\frac{j}{6}\left\{(vt)(wz)+(vw)(tz)\right\}.*$$

Now, it was found before that the symmetrical form of the Hessian is

$$H=(ab)^2a_xa_yb_ub_v-\frac{i}{6}\left\{(xu)(yv)+(xv)(yu)\right\}.$$

Multiplying this into $c_xc_uc_tc_w$, we get

$$(ab)^2(ac)(bc)\,a_yb_vc_tc_w+\frac{i}{6}\,c_yc_vc_tc_w.$$

Multiplying H into $c_xc_yc_tc_w$, we get

$$(ab)^2(ac)^2b_ub_vc_tc_w-\frac{i}{3}\,c_uc_vc_tc_w.$$

Now, H and c are symmetrical forms, and therefore, allowing for the change of y into v, we get the same result whether we join them by x, u or by x, y; we have, therefore, if f denotes the quartic,

$$(ab)^2(ac)^2b_ub_vc_tc_w-\frac{i}{3}f=(ab)^2(ac)(bc)\,a_yb_vc_tc_w+\frac{i}{6}f$$

$$=-\frac{j}{6}\left\{(uw)(tv)+(ut)(wv)\right\}+\frac{i}{6}f,$$

and therefore

$$(ab)^2(ac)^2b_ub_vc_tc_w=\frac{if}{2}-\frac{j}{6}\left\{(uw)(tv)+(ut)(wv)\right\}.\dagger$$

Fig. (36) answers to the covariant T of degorder (3, 6); it is obviously unsymmetrical, since, if we join y, z, we get (35), which does not vanish.

In getting the forms of the fourth degree we need only consider T.

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* As already remarked, Clifford makes $(ab)(ac)^2(bc)\,a_zb_vb_tc_w$ vanish identically; but this is obviously impossible, since $(vt)(wz)+(vw)(zt)$ does not vanish.

† Clifford, having made $(ab)^2(ac)(bc)\,a_yb_vc_tc_w$ vanish identically, gets

$$(ab)^2(ac)^2b_ub_vc_tc_w=-\frac{if}{2};$$

but, apart from the mistake in sign, this equation is impossible, since the left-hand side is unsymmetrical; if we make $(ab)^2(ac)^2b_ub_vc_tc_w$ symmetrical, the complementary part consists of the terms involving j in the equation in the text.

The forms are given in Figs. (37—40). Of these (37) vanishes, since it contains the vanishing graph $(ab)^3 (ac)(bc) c_x c_y$; (38) is reducible since it contains (29). The reduction in Fig. (39) is obvious, and the coefficient of (xy) is got by joining f to the reducible graph (35); (40) is got by joining (31) to itself, and therefore reduces.* We see that there are no irreducible forms of the fourth degree, and therefore none of higher degrees.

It is obvious that we might use this method in finding the form-system of any quantic; but it is also obvious that in the case of higher quantics the application of it would be exceedingly tedious, and accordingly Clifford has abandoned this method for the quintic (after finding the forms of the first, second, and third degrees), and has contented himself with taking the irreducible forms from Clebsch's *Theorie der Binären Formen.*

XI. *Theory of the Compound Form.*

The cubic has a covariant (Q) of the third order, and the quartic has a covariant (H) of the fourth order; if we take two parameters κ, λ, we can find the form-systems of the compound forms, $\kappa f+\lambda Q$, $\kappa f+\lambda H$, for the cubic and quartic respectively; the problem is, to express each form of the system in terms of κ, λ, and the form-system of f. Clebsch solves this problem by the introduction of a certain differential operator; Clifford has used a method of direct formation, which I proceed to explain; it should be mentioned that Clifford has only worked out the results for the cubic and quintic; but, as already explained, the results for the quintic are vitiated by an error. As regards the quartic, he has put down certain results of Clebsch's theory, in a way which shows that at the time he had either forgotten, or not yet noticed, that the graph for H is unsymmetrical.

In what follows, I denote the r^{th} alliance of f, ϕ by

$$(f, \phi)_r.$$

It must be remembered that

$$(f, \phi)_r = (-)^r (\phi, f)_r.$$

I denote the compound form $(\kappa f+\lambda H$ or $\kappa f+\lambda Q)$ by F. If ψ is any form appertaining to f, the corresponding form for F is denoted by ψ_F.

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* Another form of this, chosen by Clifford, reduces as shown in Fig. (41).

SEC. 1. *Cubic.*

Consider first the theory of the form

$$F = \kappa f + \lambda Q,$$

f being a cubic.

We have first of all to find Δ_F. We have

$$\Delta_F = (\kappa f + \lambda Q, \kappa f + \lambda Q)_2 = \kappa^2 (f, f)_2 + 2\kappa\lambda (f, Q)_2 + \lambda^2 (Q, Q)_2;$$

$(f, f)_2$ is, of course, Δ; $(f, Q)_2$ is given in (26). We have, if x, y are the free bonds, and R is the discriminant,

$$(f, Q)_2 = \frac{R}{2} (xy).$$

$(Q, Q)_2$ is given in (42). Now, it is obvious that, in order to join the two graphs in the way shown in the figure, one of them had to be turned round end for end; and it is easy to prove, by considering the symbolic form answering to the graph, that this operation multiplies a form by $(-)^w$, if w is the number of bonds joining the atoms of the graph, *i.e.*, the weight of the form. Therefore the graph in Fig. (42) represents [not $(QQ)_2$ but] $-(QQ)_2$; now, this graph is obviously got by joining Δ to (26) by one bond, and therefore it is $-1/2 \,.\, R \,.\, \Delta$,* and therefore we have

$$(QQ)_2 = \tfrac{1}{2} R\Delta,$$

and therefore

$$\Delta_F = \left(\kappa^2 + \frac{\lambda^2}{2} R\right) \Delta + \kappa\lambda R\,(xy) = \Theta\Delta + \kappa\lambda R\,(xy).$$

Now, the left-hand side is a quadric form; the right-hand side consists of a symmetric part, $\Theta\Delta$, and a skew part; and therefore, as we have to make all forms symmetrical, we must leave out the skew term and write

$$\Delta_F = \Theta\Delta.\dagger$$

To get Q_F, we have to join this to F by one bond. We get

$$Q_F = \Theta \left\{\Delta, \kappa f + \lambda Q\right\}_1 = \Theta \left\{\kappa (\Delta, f)_1 + \lambda (\Delta, Q)_1\right\};$$

---

* The graph is

$$\tfrac{1}{2} \,.\, R \,.\, (xz)\, \Delta_z \Delta_y = -\tfrac{1}{2} \,.\, R \,.\, \Delta_z\, (xz)\, \Delta_y = -\tfrac{1}{2} \,.\, R \,.\, \Delta_x \Delta_y \,.$$

† Clifford says,—"In any kind of multiplication $fQ = -Qf$, and therefore we have only to find the Hessian of Q." I venture to think that the reason why the term involving $\kappa\lambda$ disappears from the result is that stated in the text; the term does not necessarily disappear, but it is rejected when we make Δ_F symmetrical.

$(\Delta, f)_1$ is Q; (Δ, Q) is given in (43), and is obviously got by joining (26) to f by one bond. We have therefore

$$(\Delta, Q)_1 = -\frac{R}{2} f,$$

which gives
$$Q_F = \Theta \left\{ \kappa Q - \frac{R}{2} \lambda f \right\}.$$

R is the discriminant of Δ, and therefore

$$R_F = \Theta^2 R.$$

SEC. 2. *Quartic.*

In the theory of the quartic we consider the function

$$F = \kappa f + \lambda H.$$

We must remember that the graph of the Hessian is unsymmetrical, and that we have to use the symmetrical form (32). The Hessian of F in its ultimate* form is

$$H_F = (\kappa f + \lambda H, \kappa f + \lambda H)_2 = \kappa^2 (f, f)_2 + 2\kappa\lambda (f, H)_2 + \lambda^2 (H, H)_2.$$

The coefficient of κ^2 is, of course, H.

To find the coefficient of $2\kappa\lambda$, we take Fig. (32), and join it to f by u, v, and we get at once Fig. (44); but it was proved before that

$$(ab)^2 (ac)^2 b_x b_y c_t c_w = \frac{if}{2} - \frac{j}{2} \{(xw)(ty) + (xt)(wy)\} \quad \ldots\ldots(a),$$

and therefore, if we take the ultimate forms, we get, for Fig. (44),

$$\frac{if}{2} - \frac{if}{3} = \frac{if}{6}.$$

To find the coefficient of λ^2 we have to join H to itself by two bonds, so that we have to multiply

$$(ab)^2 a_x a_y b_u b_v - \frac{i}{6} \{(xu)(yv) + (xv)(yu)\}$$

by
$$(cd)^2 c_u c_v d_\alpha d_\beta - \frac{i}{6} \{(\alpha u)(\beta v) + (\alpha v)(\beta u)\}.$$

* The ultimate form of a quantic is what we get when we introduce scalars, and make all the sets identical; it is, in fact, the ordinary form of the quantic.

We get
$$(ab)^2(bc)^2(cd)^2 a_x a_y d_\alpha d_\beta$$
$$+\frac{i^2}{18}\{(xa)(y\beta)+(x\beta)(ya)\}-\frac{2i}{3}(ad)^2 a_x a_y d_\alpha d_\beta.$$

If we take equation (α) above, and multiply it by $d_t d_w d_\alpha d_\beta$, we get
$$(ab)^2(ac)^2(cd)^2 b_x b_y d_\alpha d_\beta = \frac{i}{2}(cd)^2 d_\alpha d_\beta c_x c_y + \frac{j}{3} d_\alpha d_\beta d_x d_y,$$

and therefore, if we take the ultimate forms, the coefficient of λ^2 is
$$\frac{iH}{2}+\frac{jf}{3}-\frac{2i}{3}H=\frac{jf}{3}-\frac{iH}{6}.$$

And therefore we get
$$H_F = \kappa^2 H + \kappa\lambda\frac{if}{3}+\lambda^2\left(\frac{jf}{3}-\frac{iH}{6}\right)$$
$$= H\left(\kappa^2-\frac{i\lambda^2}{6}\right)+f\left(\kappa\lambda\frac{i}{3}+\frac{\lambda^2 j}{3}\right)$$
$$= \tfrac{1}{3}\left(H\frac{d\Omega}{d\lambda}-f\frac{d\Omega}{d\kappa}\right),$$

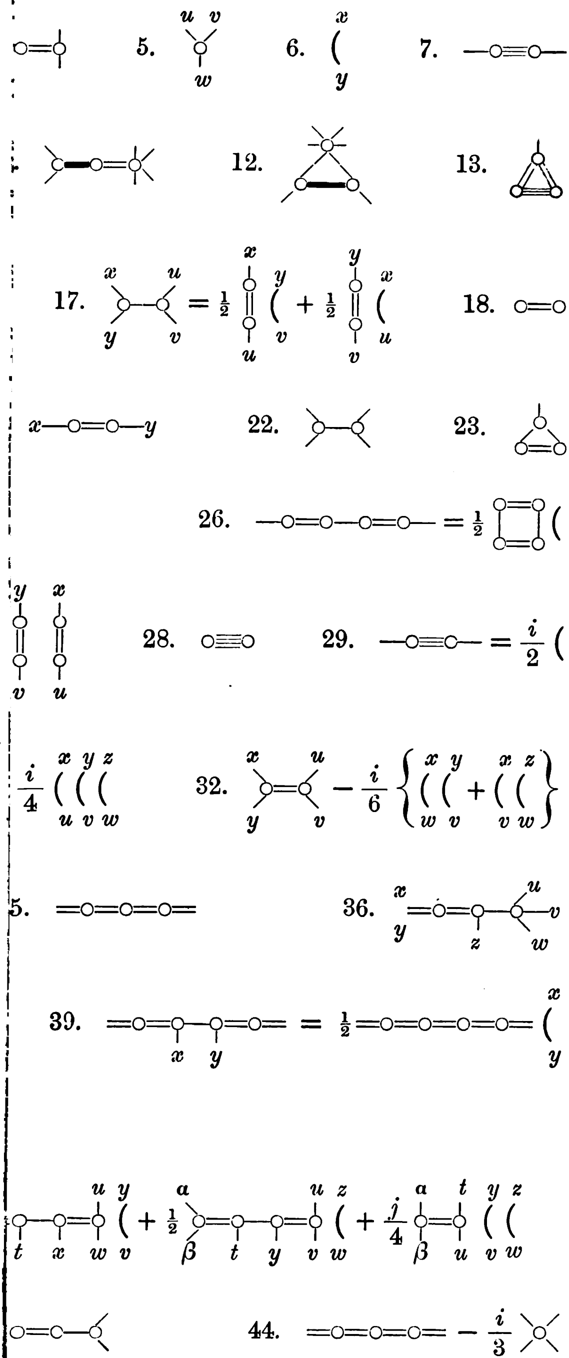
if
$$\Omega = \kappa^3 - \frac{i}{2}\kappa\lambda^2 - \frac{j}{3}\lambda^3.$$

To find T_F, we have to join this to F by one bond; we get
$$3T_F = \frac{d\Omega}{d\lambda}\{\kappa(f, H)_1+\lambda(H, H)_1\}-\frac{d\Omega}{d\kappa}\{\kappa(f, f)_1+\lambda(H, f)_1\}.$$

Now, if we take the ultimate forms, the first alliance of a quantic with itself vanishes, and we have also
$$(f, H)_1 = -(H, f)_1,$$
and therefore
$$3T_F = (f, H)_1\left(\kappa\frac{d\Omega}{d\kappa}+\lambda\frac{d\Omega}{d\lambda}\right)=3T\Omega,$$

and therefore
$$T_F = \Omega\,.\,T.$$

To find i_F, we have to find
$$(\kappa f+\lambda H, \kappa f+\lambda H)_4 = \kappa^2(f, f)_4+2\kappa\lambda(f, H)_4+\lambda^2(H, H)_4;$$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$(f, f)_4$ is i; $(H, f)_4$ is j, by definition; $(H, H)_4$ is

$$(ab)^2 (bc)^2 (cd)^2 (ad)^2 - \frac{i^2}{3};$$

but we easily find, by using equation (α) above, or by considering the graph, that

$$(ab)^2 (bc)^2 (cd)^2 (ad)^2 = \frac{i^2}{2},$$

and we therefere get

$$i_F = \kappa^2 i + 2\kappa\lambda j + \lambda^2 \frac{i^2}{6}.$$

We have $$3j_F = \left(H\frac{d\Omega}{d\kappa} - f\frac{d\Omega}{d\lambda},\ \kappa f + \lambda H\right)_4$$

$$= \frac{d\Omega}{d\kappa}\{\kappa (H, f)_4 + \lambda (H, H)_4\} - \frac{d\Omega}{d\lambda}\{\kappa (f, f)_4 + \lambda (f, H)_4\}$$

$$= \kappa j \frac{d\Omega}{d\kappa} + \frac{\lambda i^2}{6}\frac{d\Omega}{d\kappa} - \kappa i \frac{d\Omega}{d\lambda} - \lambda j \frac{d\Omega}{d\lambda},$$

and $$j_F = \frac{j}{3}\left(\kappa\frac{d\Omega}{d\kappa} - \lambda\frac{d\Omega}{d\lambda}\right) + \frac{i}{3}\left(\frac{\lambda i}{6}\frac{d\Omega}{d\kappa} - \kappa\frac{d\Omega}{d\lambda}\right)$$

$$= j\kappa^3 + \frac{i^2}{2}\kappa^2\lambda + \frac{ij}{2}\kappa\lambda^2 + \lambda^3\left(\frac{j^2}{3} - \frac{i^3}{36}\right).$$

XII. *Form-Systems.*

In this section I show how parts of Gordan's researches on form-systems, as given in Clebsch's *Binäre Formen*, can be simplified by the introduction of graphs.

1.

It will be remembered that the fundamental theorem in the theory of systems of quantics (*if two quantics have a finite form-system, then their joint system is derived from a finite form-system*) follows immediately from a lemma which can be expressed as follows :—If a power of a quantic is to be joined to any other quantic, the index of the power must not be greater than the order of the second quantic.* This is quite obvious if we consider the graphs of the two quantics. If the order of the second quantic ϕ is λ, and the index of the power of the first f is ρ, then, since the order of the alliance μ cannot be greater than λ, we are certain to satisfy all the conditions if we join

* This is not Clebsch's enunciation, but is equivalent to it.

μ f's to ϕ by one bond each, and then we are certain to have some f's left over; so that we get the product of a covariant and a power of f.

2.

In the theory of form-systems Clebsch speaks of parts of an alliance, and of the substitution of parts of an alliance for the alliance itself. This is a very simple matter if we consider graphs, for, if a graph is not symmetrical we have to make it symmetrical by adding links, and then, if we join two forms, we get, in the first instance, the graph got by joining their graphs, and then a series of terms obtained from the links. Moreover, if we join two graphs by a given number r of bonds, we can do so in various ways, since we can join any r bonds of the one to any r bonds of the other; the resulting graphs can only differ by terms derived from the complementary terms; and then it is obvious, from section (X.), that if we classify forms according to degree (in ascending order), and according to weight (in descending order), the graphs resulting from the union of two graphs by any given number of bonds can only differ by terms involving earlier forms, and that, therefore, in constructing a form-system, we can join two graphs in any way we please, provided we classify our forms in the way just described.

3.

The fundamental theorem (Clebsch's *Zerlegungssatz*) in the theory of form-systems seems much more obvious and natural if we regard it as a consequence of the following lemma:—*Every graph can be reduced to a sum of simple polygons*, where a simple polygon means an open or closed graph in which no atom is joined to more than two atoms.

For, assuming the truth of the lemma, it is obvious that in a simple polygon one of two things must happen; either all the vertices have free bonds proceeding from them, or some of the vertices are saturated; moreover, if a vertex containing an n-valent atom is saturated, it must be joined to one of the adjacent vertices by $n/2$ bonds at least; and, if the polygon was derived from an n-thic and has no saturated vertex, we can, by taking off one free bond from each vertex, get a graph derived from an $(n-1)$-thic, and we have the theorem: Every graph derived from an n-thic can be expressed as a sum of graphs, some of them derived from an $(n-1)$-thic, and the rest having one side at least containing at least $n/2$ bonds. This is the *Zerlegungssatz*.

As regards the proof of the lemma, we have only to start with the formula $(ab)(ac)(b_x c_y + b_y c_x) = (ab)^2 c_x c_y + (ac)^2 b_x b_y - (bc)^2 a_x a_y$,

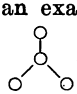
and then the lemma can be proved without any difficulty.

On the Application of Clifford's Graphs to Ordinary Binary Quantics. By A. B. Kempe, F.R.S.

[*Read Nov.* 12*th*, 1885.]

The theory of graphs, as it fell from the hands of Professor Clifford on his death, was deficient in one important particular; he had failed to show that it was directly applicable to the case of ordinary quantics. The MSS. which he left on the subject are confined to the case of binary quantics, with which alone I shall accordingly deal. By the liberality of the late Mr. Spottiswoode, who wrote an excellent account of the theory in our *Proceedings*, Vol. x., p. 204, these MSS. were reproduced in facsimile, under the title, "Mathematical Fragments, being Facsimiles of his Unfinished Papers relating to the Theory of Graphs, by the late W. K. Clifford" (London: Macmillan & Co. 1881), and copies were presented to many mathematicians in the hope that the gap might be filled up. As far as I am aware, no one has hitherto made even an attempt to do so. When presenting me with a copy of the "Fragments," Mr. Spottiswoode, knowing that the subject of the graphical representation of mathematical form was occupying my attention, expressed a wish that I should at some time do what I could in the matter. I have been too much engaged with other researches to be able to look into the question until quite recently, but have at length had the necessary opportunity.

A passage in Professor Sylvester's paper on graphs, in the *American Journal of Mathematics*, Vol. I., p. 66, line 14, at once afforded a clue to the difficulty. Clifford had overlooked the fact there pointed out, that any *co*variant $f(xy)$ may be regarded as an *in*variant of the two quantics $f(XY)$ and $Xy - Yx$, so that covariants become merged in invariants, and the variables x, y lose their distinctive character, becoming mere coefficients.

Bearing this in mind, it is at once seen that, in the notation of the method of graphs, the proper representation of an ordinary binary quantic is, taking the cubic as an example, not to be found in the symbol , but in the symbol , where the linear form ○–u is algebraically $xu_1 - yu_2$. In fact, recollecting that $u_1 u_2 = -1$, and that the same equation holds in the case of the other sets of polar quanti-

ties, we have by actual multiplication

$$(u)(v)(w)(uvw)$$

$$= (xu_1 - yu_2)(xv_1 - yv_2)(xw_1 - yw_2)(a_{111}u_1v_1w_1 + a_{112}u_1v_1w_2 + a_{121}u_1v_2w_1$$
$$+ a_{122}u_1v_2w_2 + a_{211}u_2v_1w_1 + a_{212}u_2v_1w_2 + a_{221}u_2v_2w_1 + a_{222}u_2v_2w_2)$$
$$= a_{222}x^3 + (a_{221} + a_{212} + a_{122})\,x^2y + (a_{211} + a_{121} + a_{112})\,xy^2 + a_{111}y^3\,;$$

an ordinary binary cubic.

In order to assimilate the coefficients of the cubic in this form with those of the standard form

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3,$$

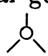
we may write

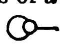
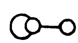
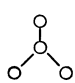
$$a_{222} = A,\ \ a_{221} = a_{212} = a_{122} = B,\ \ a_{211} = a_{121} = a_{112} = C,\ \ a_{111} = D,$$

when (uvw) becomes

$$Du_1v_1w_1 + C\,(u_1v_1w_2 + u_1v_2w_1 + u_2v_1w_1)$$
$$+ B\,(u_2v_2w_1 + u_2v_1w_2 + u_1v_2w_2) + Au_2v_2w_2,$$

a form which merely changes sign when two of the bonds u, v, w are interchanged.

This modification of the general form, which can also be made in the case of forms of any order, is more in accordance with the graphical notation than the original general form of Clifford is; for he makes the symmetrical symbol  represent the unsymmetrical form (uvw), which changes not only its sign but also its value with interchanges of u, v, w, with the result that some of his graphs are ambiguous in meaning.

It is important to note that the adoption of the symmetrical polar form causes certain graphs to vanish which do not do so when the more general form is employed. Thus every graph vanishes in which both ends of a bond proceed from the same nucleus. For example, the form  vanishes, as might be expected when it is seen that if it did not we should have the linear covariant  of the ordinary cubic 

I restate in the following sections the theory of graphs as applied to the case of ordinary binary quantics, with such additions, modifications, and definitions as may render it readily available for future use.

(I.)

Algebraical Representation of Forms.

1. Let U, U'; V, V'; W, W'; &c., be polar quantities,* and let

$$[uvw\ldots]^r = UVW'\ldots + U'VW\ldots + \ldots$$

where (1) in each term the letters $U, V, W, \ldots$ are, ignoring the accents, in the same order as the smaller ones in $[uvw\ldots]^r$, (2) there are r accented letters in each term and r only, and (3) every such term which can be formed is included in the summation.

2. We have $[uvw\ldots]^r[uvw\ldots]^s \equiv 0$ if $r+s \gtrless n$,

and $$[uvw\ldots]^r[uvw\ldots]^{n-r} \equiv (-1)^r \frac{\lfloor n}{\lfloor r \lfloor n-r}[uvw\ldots]^0[uvw\ldots]^n$$

$$\equiv (-1)^r \frac{\lfloor n}{\lfloor r \lfloor n-r} UVW\ldots U'V'W'\ldots.$$

3. Let

$$(uvw\ldots)_a = a_0[uvw\ldots]^0 + a_1[uvw\ldots]^1 + \ldots + a_r[uvw\ldots]^r + \ldots$$
$$\ldots + a_n[uvw\ldots]^n,$$

where n is the number of letters in the brackets ( ).

4. We have, if $(u)_a \equiv a_0u + a_1u' = au - a'u'$, *i.e.*, if $a_0 = a$, $a_1 = -a'$,

$$\begin{vmatrix} (u)_a & (v)_a & (w)_a & \ldots\ldots \\ (u)_\beta & (v)_\beta & (w)_\beta & \ldots\ldots \\ (u)_\gamma & (v)_\gamma & (w)_\gamma & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \;\; \ldots \end{vmatrix} \dagger$$

$$= \Sigma_{r=n}^{r=0}(-1)^r \lfloor r \; \lfloor n-r \, [uvw\ldots]^r[a\beta\gamma\ldots]^r.$$

If, then, $$(-1)^r[a\beta\gamma\ldots]^r = \frac{a_r \lfloor n}{\lfloor r \lfloor n-r},$$

the determinant is equal to $\lfloor n \; (uvw\ldots)_a$.

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* It will be observed that I do not adopt the equations

$$UU' = VV' = WW' = \ldots = 1,$$

as they lead to anomalies such as $1 = UU' \times UU' = -U^2U'^2 = 0$, and would, in Sec. 31, necessitate the unnecessary restriction $\lambda\mu' - \lambda'\mu = 1$.

† Observe that, if in this determinant rows and columns are interchanged, it will vanish identically unless it be of the first or second order.

5. We may speak of the suffixes α, β, γ ... in the preceding section as *roots* of the expression $(uvw\ldots)_\alpha$. We have

$$(u)_\alpha (u)_\beta = (\alpha\beta' - \beta\alpha')\, UU',$$

where the expression $(\alpha\beta' - \beta\alpha')$ is the same as we are familiar with in the theory of binary quantics under the form $(x_\alpha y_\beta - y_\alpha x_\beta)$, in which the pair of coordinates x_α, y_α take the place of the root α of the quantic $a(x, 1)^n$. (See Sec. 37.)

6. The symbol $(uvw\ldots)_a$ will be said to denote the *simple form* a; the suffix a being called the *mark* of the form. The letters u, v, w, &c., will be said to denote *bonds*, and the letters U, U'; V, V'; W, W'...; may be said to denote the *coordinates* of those bonds. No two simple forms can, of course, have the same marks unless they have the same number of bonds.

7. We have $\quad (\ldots u \ldots v \ldots)_a \equiv -(\ldots v \ldots u \ldots)_a,$

and therefore $\quad (\ldots u \ldots u \ldots)_a \equiv 0.$

8. The product of two or more simple forms will be termed a *compound form*.

9. The interchange of two simple forms, factors of a compound form, will not affect the value of the compound form, unless each of the simple forms contains an odd number of bonds, in which case the interchange alters the sign of the compound form. We have

accordingly $\quad (uvw\ldots)_a (uvw\ldots)_a = 0$

if the form a contains an odd number of bonds.

10. Let the two simple forms $(AB)_a$, $(AC)_b$, where the capital letters each stand for several bonds, have the m bonds A in common, and those only. Let the number of bonds in $(AB)_a$ be p, and in $(AC)_b$ be q, where $p+q=n$. Then we have

$$(AB)_a \times (AC)_b = \Sigma_{r=p}^{r=0} \Sigma_{s=q}^{s=0} a_r b_s [AB]^r \,.\, [AC]^s,\text{*}$$

and the coefficient $a_r b_s$ appears in the expansion, unless

$$[AB]^r [AC]^s = 0.$$

---

* If $D_a = a_1 \dfrac{d}{da_0} + a_2 \dfrac{d}{da_1} + a_3 \dfrac{d}{da_2} + \ldots$, we have

$$(AB)_a = \{[A]^0 + [A]^1 D_a + [A]^2 D_a^2 + \ldots\} \{a_0 [B]^0 + a_1 [B]^1 + a_2 [B]^2 + \ldots\}.$$

Now, $$[AB]^r = \Sigma_{k=r}^{k=0} [A]^{r-k} [B]^k,$$

so that $$[AB]^r [AC]^s = \Sigma_{k=r}^{k=0} \Sigma_{h=s}^{h=0} [A]^{r-k} [B]^k [A]^{s-h} [C]^h,$$

which vanishes only if $[A]^{r-k} [A]^{s-h}$ vanishes for all the possible values of k from 0 to r, and of h from 0 to s; *i.e.*, only if

$$r-k+s-h \gtrless m$$

for all of those values; *i.e.*, only if

$$r+s < m.$$

11. Hence, in the expansion of the product of the simple forms a and b, which have m bonds in common and n bonds in all, the coefficient $a_r b_s$ appears unless $r+s$ is $<m$ and $>n$.

12. Thus, if the least value of $r+s$ in the expansion of the product of the two forms is t, the number of bonds common to the two forms is also t.

13. If $f(uvw)$ be any compound form containing the bonds u, v, w alone or with others, and if the u, v, w in the symbol $f(uvw)$ refer respectively to one particular u, v, w of those occurring in the form, and not to all the u's, v's, w's, then we have the following fundamental identity,

$$f(uvw)+f(uwv)+f(vuw)+f(vwu)+f(wuv)+f(wvu) \equiv 0 \quad \ldots\text{(A)},$$

whence also $$f(uuv)+f(uvu)+f(vuu) \equiv 0 \quad \ldots\ldots\text{(B)},$$

$$f(uuu) \equiv 0 \quad \ldots\ldots\text{(C)},$$

$$f(uuvv) \equiv f(vvuu) \quad \ldots\ldots\text{(D)}.$$

14. It follows, from the identities

$$(\ldots u \ldots u \ldots)_a \equiv 0, \quad f(uuu) \equiv 0,$$

that no bond can enter into a non-vanishing simple form more than once, or into a non-vanishing compound form more than twice.

15. Any compound form in which each factor simple form has a different mark will be termed a *primary* compound form.

16. Every compound form in which some factor simple forms have the same marks may be derived from a primary compound form of the same number of factors by making certain of the different marks the same. Such a form may be termed a *degraded primary* compound form.

17. Every compound form which contains each bond twice will be the product of an ordinary quantity and the coordinates of the bonds. Such a form will be said to be *pure.* The ordinary quantity will be called the *coefficient* of the pure form.

18. There are some slight modifications of the foregoing method of representing simple forms which may be adopted with advantage in certain cases. Thus, where a compound form involves only one sort of mark a, we may suppress the suffix a in the symbol $(uvw \ldots)_a$, and write it simply $(uvw \ldots)$.

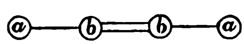
19. We may similarly suppress the suffixes in cases in which several marks are involved, provided that no two of the simple forms having different marks contain the same number of bonds. Thus, we may write $(p)_a\,(q)_a\,(puv)_b\,(quv)_b$ in the simpler form $(p)(q)(puv)(quv)$ without any ambiguity arising.

20. Again, in cases in which we have forms having different marks but the same number of bonds, we may, in lieu of using different suffixes, use different sorts of brackets; *e.g.*, we may write $(puv)_a\,(quv)_b$ thus $(puv)\ \{quv\}$.

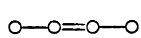
(II.)

Graphical Representation of Forms.

21. Where a compound form is considered alone, and we are not concerned with syzygetical relations, the sign of the form is immaterial, and we need not therefore trouble ourselves about the order of the bonds in the various factor simple forms, nor with the order of those simple forms in the compound form. Bearing in mind, therefore, that no bond appears more than twice, we see that the compound form is completely represented graphically by a number of small circular nuclei, which stand for the several factor simple forms, each nucleus containing the letter which is the mark of the corresponding form, lines being drawn from nucleus to nucleus representing the bonds which the corresponding forms have in common. Thus, the form $(p)_a\,(q)_a\,(puv)_b\,(quv)_b$ is fully represented by the graphical symbol



22. In cases in which we can suppress the suffixes in the algebraical representation of a form, we can also suppress them in the graphical representation. Thus, in lieu of the last figure we may have the simpler symbol



23. Again, just as in the algebraical representation we can in lieu

of suffixes use different sorts of brackets, so we can in the graphical representation use different sorts of nuclei in the place of like nuclei with different internal letters. Thus, we may represent the form

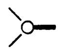
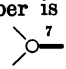
$(uv)_a\,(uv)_b$ by the symbol ○═●

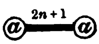
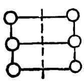
24. Compound forms which are not pure will have *free* bonds proceeding each from one nucleus only, with the other ends unattached to any nuclei; thus $(puv)(uv)$ will be represented by

–○═○

25. If both ends of a bond proceed from the same nucleus, thus , the simple form represented vanishes (Sec. 7); and therefore the compound form of which it is a factor.

26. The number of bonds of a simple form may be called its *valence.*

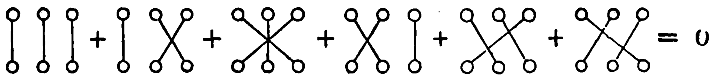
27. It will be convenient to employ a thick bond to denote an aggregation of an indefinite number of ordinary bonds, thus . Where the number is known, it may be written at the side of the thick bond, thus .

28. By Sec. 9, the form  vanishes, and this will also be the case where the forms a, a are compound forms having corresponding bonds in common. Thus, the form  vanishes.

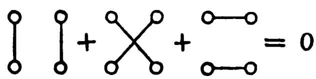
29. In the application of the graphical mode of representation to cases in which the signs of the forms must be defined, it is to be noticed that it is not necessary to assign an absolute positive or negative sign to each compound form, but only to determine the sign of each relatively to the others. Thus, the factor simple forms of a compound form may be taken in any order, as long as the same order is observed in each other compound form involving the same factor simple forms with the same or a different distribution of bonds. The same remark applies to the order in which the bonds are to be taken in determining the sign of the simple forms. It is only necessary, therefore, in a graphical representation of a compound form, to give the nuclei, and points of egress from the nuclei of the bonds, such positions on the paper that each can be separately identified; and where there are different representations of compound forms, each involving the same simple forms with the same or a different distribution of bonds, to give the corresponding nuclei and points of egress the same relative positions in each representation. As the points of egress of

each nucleus are arranged in a cycle round the nucleus, it may be as well to notice that a cyclical change of these points in the case of nuclei of odd valence does not affect the sign.

30. We have, by formula (A) of Sec. 13,



where in each term (1) the six nuclei represented may have like or unlike marks, (2) there may be any other nuclei, and (3) any other bonds, besides those represented; provided that there are no differences between the figures representing the six terms other than those due to the rearrangement of the three bonds as shown in the formula. And, by formula (B) of the same section,



where the same remark applies as in the preceding case.

(III.)

Invariance.

31. We have, if $f\,(uuvw\ldots)$ be any compound form,

$$f\,(uuvw\ldots) = f_1\,(vw\ldots)\,UU';$$

so that, if we substitute p for u where

$$P = \lambda U + \mu U',$$
$$P' = \lambda' U + \mu' U',$$

λ, μ, λ', μ' being ordinary quantities,—say, if we transform u,—we have

$$f\,(ppvw\ldots) = (\lambda\mu' - \lambda'\mu)\,f\,(uuvw\ldots),$$

so that $f\,(uuvw\ldots)$ is an invariant as regards transformations of those bonds which occur twice.

32. Let $\quad f\,(uv\ldots) = UVf_1 + UV'f_2 + U'Vf_3 + U'V'f_4;$

then we have $f\,(wu\ldots) = WUf_1 + WU'f_2 + W'Uf_3 + W'U'f_4.$

Now, if

$$V = \lambda U + \mu U',$$
$$V' = \lambda' U + \mu U',$$
$$W = \mu' U - \mu U',$$
$$W' = -\lambda' U + \lambda U',$$

we have $f(uv\ldots) = f(wu\ldots) = UU'\{\mu f_1 + \mu' f_2 - \lambda f_3 - \lambda' f_4\}$;

i.e., the transformation of one of the u's in $f(uu\ldots)$ is equivalent to the inverse transformation of the other.

33. Let $$(u)_x \equiv x_0 U + x_1 U' = xU' - yU;$$

then $$(u)_x (v)_x (w)_x \ldots = x^n [uvw\ldots]^n - x^{n-1}y\,[uvw\ldots]^{n-1} + \ldots$$
$$\ldots + (-1)^r x^{n-r} y^r [uvw\ldots]^{n-r} + \ldots + (-1)^n y^n [uvw\ldots]^0;$$

and therefore, by Sec. 2,

$$(u)_x (v)_x (w)_x \ldots (uvw\ldots)_a = [uvw\ldots]^n [uvw\ldots]^0$$
$$\times \left\{ a_0 x^n + \frac{\lfloor n}{\lfloor 1 \; \lfloor n-1} a_1 x^{n-1} y + \ldots + \frac{\lfloor n}{\lfloor r \; \lfloor n-r} a_r x^{n-r} y + \ldots + a_n y^n \right\}$$
$$= [uvw\ldots]^n [uvw\ldots]^0 a(xy)^n$$
$$= U'V'W'\ldots UVW\ldots a(xy)^n;$$

i.e., the coefficient of the pure compound form

$$(u)_x (v)_x (w)_x \ldots (uvw\ldots)_a$$

is the ordinary binary quantic $a(xy)^n$.

34. Now a linear transformation of x, y in $(u)_x$ is equivalent to the inverse transformation of u. That is, a linear transformation of x, y in $(u)_x \ldots (uvw\ldots)_a$ is equivalent to the inverse transformation of the first u, *i.e.*, by Sec. 32, to the original transformation of the last u. So that a linear transformation of x, y in $a(xy)^n$, *i.e.*, in

$$(u)_x (v)_x (w)_x \ldots (uvw\ldots)_a,$$

is equivalent to the same transformation of each of the bonds in $(uvw\ldots)_a$.

35. Hence the coefficient of every pure compound form of which the simple forms a, b, c, ... are factors, either vanishes or is an invariant of the system of quantics $a(xy)^n$, $b(xy)^m$, etc.

36. If we have two quantics $a(xy)^n$, $b(xy)^n$ of the same degree, the operator $$b_0 \frac{d}{da_0} + b_1 \frac{d}{da_1} + b_2 \frac{d}{da_2} + \ldots$$

can readily be shown to be an invariant of the two quantics; thus, if I_a is an invariant of $a(xy)^n$,

$$\left(b_0 \frac{d}{da_0} + b_1 \frac{d}{da_1} + b_2 \frac{d}{da_2} + \ldots \right) I_a$$

is an invariant of the two quantics, one degree lower in the coefficients of $a\,(xy)^n$ than I_a is. By successive applications of such operators any invariant of a system of quantics can be reduced to an invariant of a larger number of quantics, linear in the coefficients of each; each quantic in the original system, the coefficients of which are raised to the r^{th} degree in the invariant, being replaced by r similar quantics, the coefficients of which are linear in the new invariant. The invariant thus derived will be reducible, being the sum of a number of irreducible invariants derived from each other by interchanges among themselves of the quantics of the various sets which take the place of the single quantics of the original system. Each of these irreducible invariants I shall term a *primary* invariant. By regarding the various quantics of a set as one and the same quantic, we may return to the original invariant.

37. If we write $(-1)^n\,[\alpha\beta\gamma\,\ldots]^r$ for $\dfrac{a_r\,\lfloor n}{\lfloor r\,\lfloor n-r}$ (see Sec. 4), we get

$$a\,(xy)^n = [\alpha\beta\gamma\,\ldots]^n\,x^n - [\alpha\beta\gamma\,\ldots]^{n-1}\,x^{n-1}\,y + \ldots$$

$$\ldots + (-1)^r\,[\alpha\beta\gamma\,\ldots]^{n-r}\,x^{n-r}\,y^r + \ldots + (-1)^n\,[\alpha\beta\gamma\,\ldots]\,y^n$$

$$= (\alpha' x - \alpha y)(\beta' x - \beta y)(\gamma' x - \gamma y)\,\ldots \text{ to } n \text{ factors.}$$

Thus, since $(\alpha\beta' - \alpha'\beta)\;UU' = (u)_\alpha\,(u)_\beta$ (see Sec. 5), we see, from the ordinary mode of expressing invariants of quantics in terms of expressions such as $(\alpha\beta' - \alpha'\beta)$, that any primary invariant of a system of quantics $a\,(xy)^n$, $b\,(xy)^m$, ... &c. will be the quantity coefficient of the expression $\qquad I = \Sigma\,[(u)_\alpha\,(v)_\beta\,(w)_\gamma \ldots]$,

where in each term every bond occurs twice, every root occurs once, and once only, and the various terms summed are derived from each other by independent interchanges among themselves of the roots α, β, γ, ... &c. of each form a, b, ..., &c., every term so derivable being summed. But, since the roots of each form are subject to every possible interchange among themselves independently of the interchanges of the roots of the other forms, we may clearly express I as the product of a number of expressions such as

$$I_1 = \Sigma\,[(u)_\alpha\,(v)_\beta\,(w)_\gamma\,\ldots],$$

where only roots of one form occur and the terms are derived from

each other by every possible interchange of those roots, *i.e.*, where

$$I_1 = \begin{vmatrix} (u)_\alpha & (v)_\alpha & (w)_\alpha & \ldots\ldots \\ (u)_\beta & (v)_\beta & (w)_\beta & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = \underline{|n} \;\; (uvw\ldots)_\alpha,$$

(by Sec. 4). So that I is an ordinary pure primary form multiplied by a numerical factor.

38. Thus every primary invariant, and therefore every irreducible invariant, may be expressed as the coefficient of a pure compound form; *i.e.*, may be expressed by a graph.

39. Every covariant of a system of quantics $a\,(xy)^n$, $b\,(xy)^m$, &c., may be regarded as an invariant of the system of quantics $Xy-Yx$, $a\,(XY)^n$, $b\,(XY)^m$, ..., &c.; so that every invariant and covariant of a system of quantics may be represented as the coefficient of a linear function of one or more pure compound forms.

40. It is clear, from Sec. 12, how the graph for any primary invariant may be drawn. For, representing each factor simple form by a separate nucleus, the rule of that section shows how many bonds each pair has in common, and thus the graph is given.

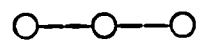
(IV.)

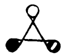
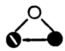
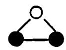
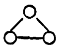
Some Forms.

41. *Forms with univalent factors only.* The invariant of two univalent forms is ○–●, which vanishes if the forms are the same. An invariant of any even number of univalent forms which contains each form only once will be the product of a number of invariants such as ○–●. We may, of course, have invariants of an odd number of forms, if some of the forms occur more than once. The various invariants which can be composed with a number of univalent forms are connected syzygetically by equations such as those in Sec. 30.

42. In the subsequent sections the form ○– will always be supposed to have the mark x, so that the coefficients of forms involving it are all covariants. Further, when quadrics, cubics, &c., and their invariants and covariants are spoken of, it is to be understood that reference is made to forms the coefficients of which are ordinary quadrics, cubics, &c., and their invariants and covariants respectively.

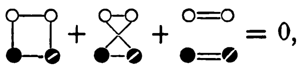
43. *Forms with bivalent factors only.* The invariant of two bivalent forms each of which occurs once only is ○═●. The two forms may be alike, in which case we get the discriminant ○═○ of the quadric

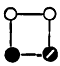
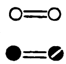
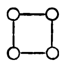


44. We have  $= -$  $= -$ ; so that  $= 0$, and therefore also  $= 0$.

45. We may, in a similar way, show that a circuit containing any odd number of factors having the same marks vanishes.

46. We have, by Sec. 30,

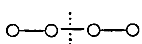


i.e., we have  $= -\frac{1}{2}$ , whence  $= -\frac{1}{2}$ ○═○^{2}.

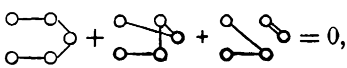
47. And generally, a circuit containing an even number $2n$ of bivalent forms of the same sort

$$= (-\tfrac{1}{2})^{n-1}\ \text{○═○}^n.$$

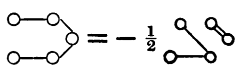
48. *Forms with bivalent and univalent factors only, each of one sort.* The form ○—○—○ is the quadric. Any form with two univalent factors and an even number of bivalent factors vanishes; thus



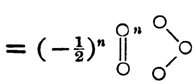
vanishes (see Sec. 28). We have



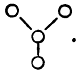
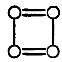
i.e.,

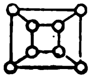
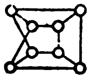
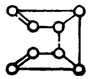


And, generally, any chain with two univalent terminals and an odd number $2n+1$ of intermediate bivalent factors

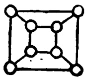
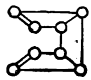


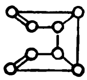
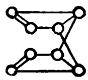
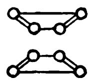
49. *Forms with trivalent factors of one sort only, i.e.,* invariants of

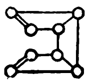
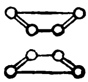
the cubic  . We have  $= 0$. The form  is the discriminant. Every other form is either a power of this multiplied by a numerical coefficient, or else vanishes. For example, we have

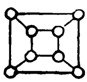
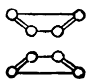
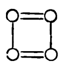
 +  +  $= 0$.

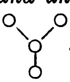
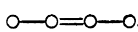
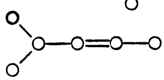
Now, if the figure which is the middle term of the left hand member be turned clockwise through a right angle, we get a new figure of the same value and sign as the original one; and, if in the new figure the two nuclei at the left-hand top corner be transposed, we return to the original figure, but the transformation changes the sign; thus the middle term vanishes, and we have

 +  $= 0$.

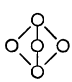
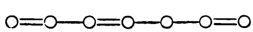
But  +  +  $= 0$,

i.e. 2  +  $= 0$,

hence 2  =  =  ^{2}.

50. *Forms with trivalent and univalent factors only, each of one sort, i.e.*, covariants of the cubic  . The hessian is . The cubic covariant is . By successive applications of the second formula of Sec. 30, the relation $J^2 - DU^2 + 4H^3 = 0$ can be shown without any difficulty.

51. *Forms with trivalent and bivalent factors only, each of one sort, i.e.*, invariants of a quadric and cubic. (See, with reference to this and the next three Sections, Salmon's *Modern Higher Algebra*, 4th Edition, pages 187 ff).

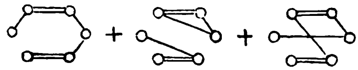
$I =$ , $R =$ , $M =$ .

52. *Forms with trivalent, bivalent, and univalent factors only, each of one sort, i.e.*, covariants of a quadric and cubic.

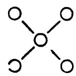
$L_1 =$ ○=○—○, $L_2 =$ ○=○—○—○, $L_3 =$ ○=○—○=○—○,

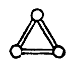
$L_4 =$ ○=○—○=○—○—○.

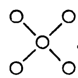
We have also the form ○=○—○—○=○—○, but

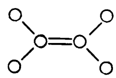
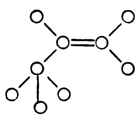
 $= 0$;

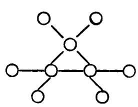
and similarly in other cases.

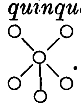
53. *Forms involving quadrivalent factors of one sort only, i.e.*, invariants of the quartic 

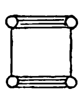
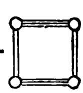
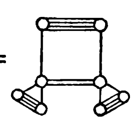
$S =$ ○≡○, $T =$ 

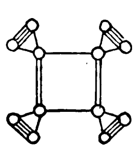
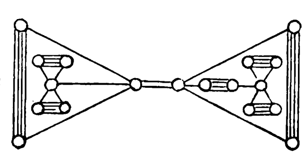
54. *Forms involving quadrivalent and univalent factors only, each of one sort, i.e.*, covariants of the quartic .

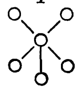
$H =$ , $J =$ 

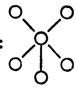
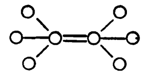
The form  vanishes.

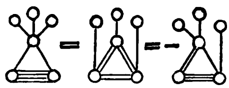
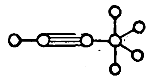
55. *Forms involving quinquevalent factors of one sort only, i.e.*, invariants of the quintic . The numbers attached to the forms in this and the following Section are those of page 237 of Salmon's *Modern Higher Algebra.*

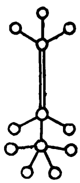
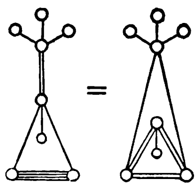
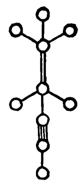
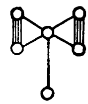
(7) $J =$  $= -$  (17) $K =$ 

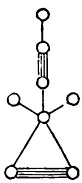
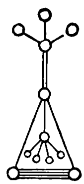
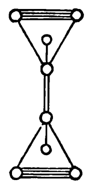
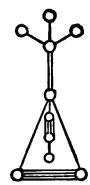
(21) $L =$  23) $I =$ 

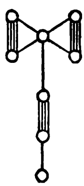
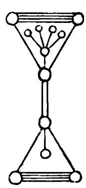
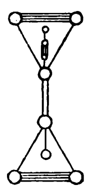
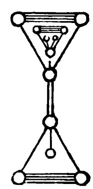
56. *Forms involving univalent and quinquevalent factors only, each of one sort, i.e.,* covariants of the quintic 

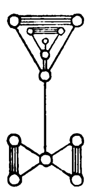
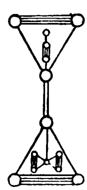
(1) $U =$  (2) $S =$  (3) $H =$ 

(4) $T =$  (5) 

(6)  (8)  (9)  (10) 

(11)  (12)  (13)  (14) 

(15)  (16)  (18)  (19) 

(20)  (22) 

Thursday, December 10th, 1885.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Mr. A. E. Haynes, M.Ph., Hillsdale College, Michigan, was elected a member.

The Auditor (Mr. Basset) made his report; a vote of thanks was unanimously accorded to him for his services. The Treasurer's report was then adopted.

The following communications were made :—

On the Numerical Solution of Cubic Equations: G. Heppel, M.A.

On a Theorem in Kinematics: J. J. Walker, F.R.S.

Note on the Induction of Electric Currents in an Infinite Plane Current Sheet, which is rotating in a Field of Magnetic Force: A. B. Basset, M.A.

The following presents were received :—

"The Nautical Almanac" for 1889.

"Educational Times," for December.

"Proceedings of the Cambridge Philosophical Society," Vol. v., Part iv.; Camb. 1885.

"Proceedings of the Royal Irish Academy,"—"Science," Ser. ii., Vol. iv., Nos. 3 and 4; "Polite Literature and Antiquities," Ser. ii., Vol. ii., No. 6.

"Transactions of the Royal Irish Academy,"—"Science," Vol. xxviii., Nos. 17, 18, 19, and 20.

"Royal Irish Academy,"—"Todd Lecture Series," Vol. ii., Part i.; "Irish Lexicography," an Introductory Lecture, by R. Atkinson, M.A., LL.D; 8vo, Dublin, 1885.

"Johns Hopkins University Circulars," Vol. v., No. 43.

"Smithsonian Report," for 1883; Washington, 1885.

"Bulletin des Sciences Mathématiques," 2e Ser., T. ix., Dec. 1885.

"Beiblatter zu den Annalen der Physik und Chemie," B. ix., St. 11.

"Atti della R. Accademia dei Lincei," Vol. i., Fasc. 24 and 25.

"Archives Néerlandaises des Sciences Exactes et Naturelles," T. xx., L. 3.

"Jahrbuch über die Fortschritte der Mathematik," xv. 1, Jahrgang 1883.

"Cours de Mécanique," par M. Despeyrous, avec des Notes par M. G. Darboux. Tome Second; 8vo, Paris, 1886.

"Bibliographie Néerlandaise Historique-Scientifique des Ouvrages Importants dont les Auteurs sont nés aux 16°, 17°, et 18° siècles, sur les Sciences Mathématiques et Physiques avec leurs applications," par le Dr. D. Bierens de Haan; 4to, Rome, 1883. "Nouvelles Additions," 4to: from Dr. Bierens de Haan. ("Extrait du Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche," Tomo xiv., 1882; xv., 1882; xvi., 1883.)

"Nieuw Archief voor Wiskunde," D. xii., St. 1; Amsterdam, 1885: from Dr. Bierens de Haan.

"Derde Rapport van de Huygens-Commissie," 8vo; Amsterdam, 1885: from Dr. Bierens de Haan.

"Bibliografia Achille Sannia — Lezioni di Geometria Projettiva dettate nella Università di Napoli." (Presented with the "Giornale di Matematica," July, August, 1885.)

"United States Coast and Geodetic Survey," J. E. Hilgard, Superintendent,—"Methods and Results," Appendix No. 15, Report for 1884, "Gravity Research, use of the Noddy for measuring the Swaying of a Pendulum support;" Appendix No. 16, Report for 1884, "Gravity Research, effect of the Flexure of a Pendulum upon its period of Oscillation;" Washington, 1885.

On a Theorem in Kinematics. By Mr. J. J. WALKER.

[*Read Dec.* 10*th*, 1885.]

The principle on which the composition of rotations, of any magnitudes, about axes in space, is founded, is Hamilton's Theorem (*Proc. R. I. A.*, 1834), obtained and demonstrated* by the then new Quaternion method—that two successive rotations through angles 2θ, $2\theta'$ about two intersecting axes OA, OB, regarded as radii of a sphere, are equivalent to a single rotation, through an angle 2ϕ, about a third radius OC, determined by making the angles BAC, ABC of the spherical triangle ABC equal to θ, θ' respectively; ϕ being equal to the supplement of the third angle ACB of the triangle.

I have recently noticed the following connected theorem, which seems of interest in itself, and the geometrical proof of which contains a demonstration of Hamilton's theorem; viz., In the successive rotations, of any magnitude, of a rigid body about two intersecting axes, any line in the body whose direction passes through their intersection describes portions of two right cones: *these meet again in a side which is common also to the cone of the equivalent single rotation.*

Let α, β, ρ be the unit vectors OA, OB, OR, the last rotating about the first two through angles 2θ, $2\theta'$ respectively, so as to coincide with ρ_1, ρ_2 (OR_1, OR_2) successively; also, let γ be OC, the axis of the single equivalent rotation 2ϕ, which shall at once make ρ coincide with ρ_2. If the cones on which ρ moves in its rotations, and of which ρ_1 is one common side, meet again in σ (OS), and the angle $R_1AS = 2\psi$, then

$$\sigma = \rho + 2\sin(\theta+\psi)\left\{V\alpha\rho\cos(\theta+\psi) + \alpha V\alpha\rho\sin(\theta+\psi)\right\}\dagger \quad \ldots(1),$$

---

* Since verified by conceiving a pyramid to roll round another fixed equal pyramid, which is the reflexion or image of the first relative to their common face.

† This form of σ expresses that it is the resultant of ρ and a vector equal and parallel to the chord of the arc described by the term of ρ in rotating.

and, as the condition that this vector shall lie on the right cone about γ as axis and containing ρ as a side, it is necessary that

$$S\gamma\sigma = S\gamma\rho,$$

i.e. (1), $$S\gamma\left\{V\alpha\rho\cos(\theta+\psi)+\alpha V\alpha\rho\sin(\theta+\psi)\right\} = 0 \quad \text{.........(2).}$$

Now, $$\gamma\sin\phi = \alpha\sin\theta\cos\theta'+\beta\sin\theta'\cos\theta+V\beta\alpha\sin\theta\sin\theta',$$

and, since the substitution of α, multiplied by any scalar, plainly satisfies (2), it is to be shown that

$$(S.\beta V\alpha\rho\cos\theta+S.V\beta\alpha V\alpha\rho\sin\theta)\cos(\theta+\psi)$$
$$+(S.\beta\alpha V\alpha\rho\cos\theta-S.\beta V\alpha\rho\sin\theta)\sin(\theta+\psi) = 0 \text{...(3).}$$

Let r, c be the angles which ρ, β make with α; then, since

$$S.\beta\alpha V\alpha\rho = S.V\beta\alpha V\alpha\rho = -S.V\alpha\beta V\alpha\rho = \sin c\sin r\cos(2\theta+\psi),$$

$2\theta+\psi$ being the angle BAR between the planes of $\alpha\rho$, $\alpha\beta$, the expression (3) reduces to

$$\cos(2\theta+\psi)\left\{S.\beta V\alpha\rho+\sin c\sin r\sin(2\theta+\psi)\right\},$$

which vanishes, $V\alpha\rho$ being the vector (δ) to the pole (D) of AR multiplied by $\sin r$, and $S\beta\delta$ being equal to

$$-\cos BD = -\sin AB\cos BAD = -\sin AB\sin BAR$$
$$= -\sin c\sin(2\theta+\psi).$$

It may be convenient, to avoid reference, to verify here that γ has the vector value assigned in the above proof. It is the vector which remains unchanged by the two rotations round α, β; *i.e.*,

$$\gamma = \beta^{\frac{\theta'}{\pi}}\alpha^{\frac{\theta}{\pi}}\gamma\alpha^{-\frac{\theta}{\pi}}\beta^{-\frac{\theta'}{\pi}},$$

or $$\gamma\beta^{\frac{\theta'}{\pi}}\alpha^{\frac{\theta}{\pi}} = \beta^{\frac{\theta'}{\pi}}\alpha^{\frac{\theta}{\pi}}\gamma,$$

or $$V.\gamma V\beta^{\frac{\theta'}{\pi}}\alpha^{\frac{\theta}{\pi}} = 0,$$

whence $$\gamma V\beta^{\frac{\theta'}{\pi}}\alpha^{\frac{\theta}{\pi}} = \text{a scalar},$$

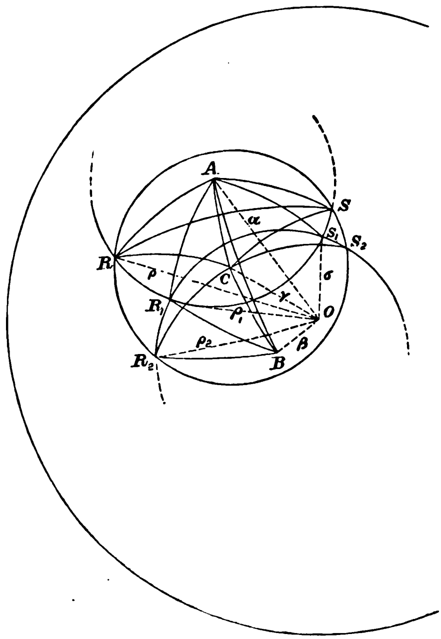
i.e., $$V\beta^{\frac{\theta'}{\pi}}\alpha^{\frac{\theta}{\pi}} = \gamma \text{ (to a scalar factor)},$$

or $$k\gamma = \beta\sin\theta'\cos\theta+\alpha\sin\theta\cos\theta'+V\beta\alpha\sin\theta\sin\theta',$$

and the particular value of k, which may be easily verified to be $\sin\phi$, does not concern the application just made.

Having thus verified the theorem of the cointersection of the three cones by the Quaternion method, I proceed to show how it, and the original theorem, may be derived from the elementary geometry of the sphere.

Let a radius OR of a sphere rotate about another OA into the position OR_1, from which it is carried by a second rotation about another radius OB into the position OR_2. Let the small circles, arcs of which RR_1, R_1R_2 are described in the above rotations, meet again in S_1; and finally, let any third small circle, about C as pole, be described through R, R_2 cutting the first pair again in S, S_2. Joining the



points on the sphere by arcs of great circles, as in the figure, the angles referred to being dihedral angles contained by planes of great

circles, $\angle RAS = 2\angle CAS, \quad \angle R_1AS_1 = 2\angle BAS_1,$

therefore
$$\angle RAR_1 + \angle SAS_1 = 2\angle CAS - 2\angle BAS_1$$
$$= 2\angle BAC + 2\angle SAS_1,$$

i.e.,
$$\angle RAR_1 - \angle SAS_1 = 2\angle BAC.$$

Similarly,
$$\angle R_1BR_2 - \angle S_1BS_2 = 2\angle ABC.$$

Also $$\angle RCS = 2\angle ACS, \quad \angle R_2CS_2 = 2\angle BCS_2,$$

therefore $$2\pi - \angle RCS - \angle R_2CS_2,$$

or $$\angle RCR_2 + \angle SCS_2 = 2(\pi - \angle ACB + \angle SCS_2),$$

i.e., $$\angle RCR_2 - \angle SCS_2 = 2(\pi - \angle ACB).$$

If the point S_1 fell outside the circle through RR_2 of which C is pole, the signs connecting the pairs of angles in the foregoing results would change; viz., they would be

$$\angle RAR_1 + \angle SAS_1 = 2\angle BAC, \quad \angle R_1BR_2 + \angle S_1BS_2 = 2ABC,$$

$$\angle R_1CR_2 + \angle S_1CS_2 = 2(\pi - \angle BCA).$$

But, if the circle through RR_2 about C as. pole passed through S_1, viz., if SS_1S_2 coincided in S, then

$$\angle BAC = \angle RAR_1, \quad \angle ABC = \angle R_1BR_2, \text{ and } \angle R_1CR_2 = 2(\pi - \angle BCA),$$

so that in this case, and this only, would the axis OC and $\angle R_1CR_2$ be the same for all radii whatever; *i.e.*, would the whole body, and not merely a given vector or line in it, perform a rotation about OC equivalent to the two of magnitudes 2θ, $2\theta'$ about OA, OB successively; and that rotation would be through an angle equal to $2(\pi - \angle ACB)$.

The axis (OC) of the resultant rotation, in this treatment of Hamilton's theorem, is thus fixed in a manner wholly free from any ambiguity; viz., it is the connector of the poles of the small circle determined, on any sphere about the fixed point O as centre, by the initial (OR) and final (OR_2) positions of a radius, and the reflexion (OS) of its intermediate position (OR_1) relatively to the plane of the axes (OA, OB) of the component rotations.

Thursday, January 14th, 1886.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Mr. J. B. Colgrove, M.A., Loughborough Grammar School, was elected a member; and Mr. S. O. Roberts was admitted into the Society.

The President (Mr. Walker, Vice-President, in the Chair) communicated a question, proposed by Mr. H. M. Taylor, as to the novelty of an elementary result in Plane Conics, and the extension of the property to other curves; several members made suggestions as to possible sources of information. The President having resumed the chair, Mrs. Bryant read a paper on "Logarithms in General Logic"; a long discussion ensued, in which Mr. Kempe, Prof. Sylvester, the President, Mr. S. Roberts, and Mrs. Bryant, took part.

Mr. J. Hammond (Prof. Sylvester, Vice-President, in the Chair) read a paper "On a Class of Integrable Reciprocants." Captain Macmahon made a short communication on a Certain Property of Seminvariants.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xxxix., No. 239.

"Educational Times," for January.

"Mathematical Papers, chiefly connected with the q-Series in Elliptic Functions," 1883—1885, 8vo; by J. W. L. Glaisher, F.R.S.; Cambridge, 1885.

"Manchester Literary and Philosophical Society—Memoirs," Vol. viii., Third Series, 8vo, London, 1884; "Proceedings," Vols. xxiii. and xxiv., 8vo, Manchester, 1884 and 1885.

"On the Discovery of the Periodic Law, and on Relations among the Atomic Weights," by J. A. R. Newlands; 8vo, London, 1884.

"Johns Hopkins University Circulars," Vol. v., No. 45.

"Report of the National Academy of Sciences," for the years 1883 and 1884; 8vo, Washington, 1884 and 1885.

"Proceedings of the National Academy of Sciences," Vol. i., Part ii., 8vo; Washington, 1884.

"Memoirs of the National Academy of Sciences," Vol. iii., Part i., 4to, 1884; Washington, 1885.

"Beiblätter zu den Annalen der Physik und Chemie," B. ix., St. 12.

"Jornal de Sciencias Mathematicas e Astronomicas," publicado pelo Dr. F. Gomes Teixeira, Vol. vi., No. 4; Coimbra, 1885.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. i., F. 26 and 27.

"Journal für Mathematik," Bd. xcix., Heft 3.

"Les Fonctions d'une Seule Variable à un Nombre quelconque de Périodes," par F. Casorati.

On a Class of Integrable Reciprocants. *By* Mr. J. HAMMOND.

[*Read January* 14*th*, 1886.]

1. The notation, and nomenclature, of the present paper is that of Prof. Sylvester's Inaugural Lecture (*Nature*, Jan. 7th, 1886); in which $\frac{dy}{dx}$ is denoted by the single letter t, and $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots$ by $a, b, c, \ldots$ respectively, and a Reciprocant is a function of $t, a, b, c, \ldots$ which, to a factor *près*, remains unaltered when the variables x and y are interchanged.

Or, if $$\frac{dx}{dy} = \tau,\quad \frac{d^2x}{dy^2} = \alpha,\quad \frac{d^3x}{dy^3} = \beta,\quad \frac{d^4x}{dy^4} = \gamma, \ \ldots\ldots$$

and $$\phi(\tau, \alpha, \beta, \gamma, \ldots) = \phi(t, a, b, c, \ldots) \text{ to a factor } pr\grave{e}s,$$

the function ϕ is a reciprocant.

The class of reciprocants referred to in the title is characterised by having an integral of the form

$$a = F(t) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

i.e., this is an integral of the differential equation

$$\phi(t, a, b, c, \ldots) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

obtained by equating the reciprocant ϕ to zero.

The integration of (1) may be performed by well-known and simple methods which show that, in the case considered, the complete primitive of (2) is the equation of a curve whose Cartesian coordinates are

$$\left.\begin{aligned} x &= \int \frac{dt}{F(t)} + \text{const.} \\ y &= \int \frac{t\,dt}{F(t)} + \text{const.} \end{aligned}\right\} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

or whose intrinsic equation is

$$s = \int \frac{\sec^3 \psi \, d\psi}{F(\tan \psi)} + \text{const.}$$

2. The first reciprocant of this class that came under the author's notice was Prof. Sylvester's orthogonal reciprocant, which is the

left-hand member of the equation,*

$$(1+t^2)\,c-10abt+15a^3=0\quad\text{.......................(4).}$$

This may be solved by assuming its first integral to be of the form

$$Pa^2+Qb=\text{const.}\quad\text{.........................(5),}$$

where P and Q are unknown functions of t in which neither a, b, c, ... nor the variables x, y enter.

Then, since
$$\frac{d}{dx}=a\delta_t+b\delta_a+c\delta_b+\dots,$$

the differentiation of (5) gives

$$\frac{d}{dx}(Pa^2+Qb)=a^3\frac{dP}{dt}+ab\left(2P+\frac{dQ}{dt}\right)+cQ=0.$$

Comparing this with (4), we see that

$$Q:2P+\frac{dQ}{dt}:\frac{dP}{dt}=1+t^2:-10t:15\quad\text{..............(6),}$$

whence P and Q may be determined.

But it will shorten the work to assume

$$\left.\begin{aligned}P&=\frac{d^4u}{dt^4}\\ Q&=\frac{1}{15}(1+t^2)\frac{d^5u}{dt^5}\end{aligned}\right\}\quad\text{..........................(7),}$$

when one of the equations (6) is satisfied, and the other becomes

$$(1+t^2)\frac{d^6u}{dt^6}+12t\frac{d^5u}{dt^5}+30\frac{d^4u}{dt^4}=0,$$

which, written in the form

$$\frac{d^6}{dt^6}\{(1+t^2)\,u\}=0,$$

---

* Captain MacMahon has transformed (4) into

$$\frac{d^3\psi}{ds^3}+18\left(\frac{d\psi}{ds}\right)^3=0,$$

which is the differential equation of the curve whose intrinsic equation is

$$\sin 3\psi=\frac{1}{\sqrt{2}}\,\mathrm{sn}\left(3m\sqrt{2s},\ \frac{1}{\sqrt{2}}\right).$$

gives
$$u = \frac{A+Bt}{1+t^2} + C + Dt + Et^2 + Ft^3.$$

It follows, from (7), that we may take

$$\left.\begin{aligned} P_1 &= \frac{d^4}{dt^4}\left(\frac{1}{1+t^2}\right) \\ Q_1 &= \frac{1}{15}(1+t^2)\frac{d^5}{dt^5}\left(\frac{1}{1+t^2}\right) \\ P_2 &= \frac{d^4}{dt^4}\left(\frac{t}{1+t^2}\right) \\ Q_2 &= \frac{1}{15}(1+t^2)\frac{d^5}{dt^5}\left(\frac{t}{1+t^2}\right) \end{aligned}\right\} \quad \text{.....................(8),}$$

and thus obtain two first integrals of the form (5),

$$P_1 a^2 + Q_1 b = C_1,$$

$$P_2 a^2 + Q_2 b = C_2,$$

from which, by the elimination of b, a is found expressed as a function of t, thus

$$a = \sqrt{\frac{C_1 Q_2 - C_2 Q_1}{P_1 Q_2 - P_2 Q_1}} = F(t) \quad \text{.....................(9),}$$

and the complete primitive is of the same form as (3), containing four arbitrary constants; two of which are the C_1 and C_2 which appear in the function of t just found, the remaining two being the constants of integration in (3).

3. Writing
$$t = \tan\theta,$$

and performing the differentiations indicated in (8), we find, without difficulty,

$$P_1 = 24\cos^5\theta\cos 5\theta,$$

$$Q_1 = -8\cos^4\theta\sin 6\theta,$$

$$P_2 = 24\cos^5\theta\sin 5\theta,$$

$$Q_2 = 8\cos^4\theta\cos 6\theta,$$

whence
$$P_1 Q_2 - P_2 Q_1 = 192\cos^{10}\theta,$$

and these values, substituted in (9), will give

$$a = \sec^3\theta\sqrt{\kappa\cos 6\theta + \lambda\sin 6\theta} \quad \text{.....................(10),}$$

in which κ and λ are mere numerical multiples of the former arbitrary constants C_1 and C_2.

Now (10) is the second differential equation of the curve represented by the complete primitive of (4), and may be written in the form

$$\rho^2 \cos 6\,(\theta - A) = B \qquad\text{.........................(11)},$$

where $\rho = \dfrac{\sec^3\theta}{a}$ is the radius of curvature of this curve at any point, and θ is the inclination of the tangent at that point to the axis of x.

Thus, if we refer the curve to new axes, making an angle A with the old ones, and take for our unit of linear measurement the length of the radius of curvature which is parallel to the new axis of y, we may write

$$\rho^2 \cos 6\theta = 1,$$

whence we obtain

$$s = \int \frac{d\theta}{\sqrt{\cos 6\theta}} = \int \frac{d\theta}{\sqrt{1-2\sin^2 3\theta}},$$

leading to the intrinsic equation to the curve

$$\sin 3\theta = \frac{1}{\sqrt{2}}\,\mathrm{sn}\left(3\sqrt{2}s,\ \frac{1}{\sqrt{2}}\right).$$

Comparing (10) with (11), we see that

$$\kappa = \frac{\cos 6A}{B},\quad \lambda = \frac{\sin 6A}{B},$$

so that the corresponding simplification of (10) is effected by writing $\kappa = 1$ and $\lambda = 0$. The Cartesian coordinates of the curve are easily seen to be

$$x = \int \frac{\cos\theta\, d\theta}{\sqrt{\cos 6\theta}},\quad y = \int \frac{\sin\theta\, d\theta}{\sqrt{\cos 6\theta}},$$

and the Cartesian equation of the curve is found by eliminating θ between these two in the following manner :—By means of the first $\cot^2\theta$ is expressible as an elliptic function of x, and by means of the second $\tan^2\theta$ is expressible as an elliptic function of y; the product of these two elliptic functions equated to unity is the Cartesian equation of the curve.

In fact, if we write

$$\tan^2\theta = \sin^2\phi + \frac{k'^2}{k^2}\cos^2\phi,$$

where

$$k = \sin 15^\circ \text{ and } k' = \sin 75^\circ,$$

after some easy reductions we shall find

$$my = \int \frac{d\phi}{\sqrt{1-k'^2 \sin^2 \phi}},$$

where $$m^2 = 8\sqrt{3},$$

so that $$\tan^2\theta = \mathrm{sn}^2(my, k') + \frac{k'^2}{k^2}\mathrm{cn}^2(my, k'),$$

and similarly $$\cot^2\theta = \mathrm{sn}^2(mx, k) + \frac{k^2}{k'^2}\mathrm{cn}^2(mx, k).$$

Thus the curve is similar to the one whose equation is

$$1 = \left\{\mathrm{sn}^2(x, k) + \frac{k^2}{k'^2}\mathrm{cn}^2(x, k)\right\}\left\{\mathrm{sn}^2(y, k') + \frac{k'^2}{k^2}\mathrm{cn}^2(y, k')\right\},$$

which reduces to $$k'^2 \mathrm{tn}^2(x, k) = k^2 \mathrm{tn}^2(y, k').$$

The complete primitive of (4), with its full number of arbitrary constants, may be obtained from this equation by the *orthogonal* substitution of $$lx + my + n_1 \text{ for } x,$$

and $$mx - ly + n_2 \text{ for } y;$$

as may be verified by differentiating it four times in succession, after the substitution has been made, and eliminating the four arbitrary constants l, m, n_1, n_2.

For the results given in the present article I am indebted to Prof. Greenhill, who first pointed out the advantage of using $\tan\theta$ instead of t. The restoration of t in (10) will give

$$a^2 = \kappa(1 - 15t^2 + 15t^4 - t^6) + \lambda(6t - 20t^3 + 6t^5) \quad \text{.........(12)},$$

leading to the form of the complete primitive of (4), originally given in *Nature*, viz.,

$$\left.\begin{aligned} x &= \int \frac{dt}{\sqrt{\kappa(1-15t^2+15t^4-t^6)+\lambda(6t-20t^3+6t^5)}} + \mu \\ y &= \int \frac{t\,dt}{\sqrt{\kappa(1-15t^2+15t^4-t^6)+\lambda(6t-20t^3+6t^5)}} + \nu \end{aligned}\right\},$$

where the integrals which occur are reducible to elliptic, instead of being, as was stated without due consideration, hyper-elliptic integrals.

4. The form of (12) clearly indicates that it is an integral of a

reciprocant; for, on interchanging the dependent and independent variables, or, what is the same thing, writing

$$a = -\frac{\alpha}{\tau^3} \quad \text{and} \quad t = \frac{1}{\tau},$$

$$a^2 = \kappa\,(1-15t^2+15t^4-t^6)+\lambda\,(6t-20t^3+6t^5)$$

becomes $$\alpha^2 = \kappa'\,(1-15\tau^2+15\tau^4-\tau^6)+\lambda'\,(6\tau-20\tau^3+6\tau^5),^*$$

and obviously the differential equation obtained by differentiating the latter twice with respect to y, and eliminating κ' and λ', will be precisely similar to that found by differentiating the former twice with respect to x, and eliminating λ and μ; *i.e.*, it will be precisely similar to the original equation (4).

More generally, if, when the variables are interchanged,

$$F(t,\, a,\, A,\, B,\, C,\, \ldots) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

becomes $$F(\tau,\, \alpha,\, A',\, B',\, C',\, \ldots) = 0,$$

the form of the function remaining unaltered, and only the values of the arbitrary constants A, B, C, ... suffering change; the same reasoning as before will show that (13) is an integral of a reciprocant.

And if the same permanence of form accompanies any linear substitution of the variables, say

$$\left.\begin{aligned} x &= lX+mY+n \\ y &= l'X+m'Y+n' \end{aligned}\right\},$$

(13) will be an integral of what, after Prof. Sylvester, we call a Pure Reciprocant. In this case τ and α are defined by

$$\tau = \frac{dY}{dX}, \quad \alpha = \frac{d^2Y}{dX^2};$$

and, since $\quad dx = (l+m\tau)\,dX$ and $dy = (l'+m'\tau)\,dX,$

we have $$t = \frac{l'+m'\tau}{l+m\tau} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

Now, writing $$a = \frac{dt}{dx} = \frac{1}{l+m\tau}\,\frac{dt}{dX},$$

---

* In the case considered, $\kappa' = -\kappa$ and $\lambda' = \lambda$; but the form of the relation between a and t is also permanent when subjected to any *orthogonal* transformation, in which case the relations between κ, λ and κ', λ' will differ from those given.

we obtain, from (14), $\alpha = \dfrac{lm'-l'm}{(l+m\tau)^3}\,a$ ....(15),

and obviously

$$(A, B, C, \ldots \between 1, t)^\kappa = \frac{1}{(l+m\tau)^\kappa}(A', B', C', \ldots \between 1, \tau)^\kappa,$$

so that the relation $a^{\frac{1}{3}\kappa} = (A, B, C, \ldots \between 1, t)^\kappa$ ....(16)

possesses this permanence of form, and is consequently an integral of some pure reciprocant.

5. From (16), by making $\kappa = 1, 2, 3 \ldots$ in succession, we derive, by a process of alternate differentiation with respect to x and division by a, a series of pure reciprocants, from which, as Protomorphs, all other pure reciprocants may be algebraically deduced. The degree of these is however, in general, greater than that of Prof. Sylvester's series of Protomorphs.

Thus, when $\kappa = 1$, we have

$$a^{\frac{1}{3}} = A + Bt,$$

whence, by differentiation with respect to x,

$$\tfrac{1}{3}a^{-\frac{2}{3}}b = Ba;$$

dividing by a and differentiating again with respect to x, we have

$$\frac{d}{dx}(a^{-\frac{5}{3}}b) = a^{-\frac{5}{3}}c - \tfrac{5}{3}a^{-\frac{8}{3}}b^2 = 0,$$

or, $$3ac - 5b^2 = 0,$$

where the expression on the left is Prof. Sylvester's Parabolic Protomorph.

In exactly the same way, the Mongian

$$9a^2d - 45abc + 40b^3$$

is obtained from $$a^{\frac{2}{3}} = (A, B, C \between 1, t)^2.$$

But, when $\kappa = 3$,

$$a = (A, B, C, D \between 1, t)^3$$

gives the form $$a^3e - 7a^2bd - 4a^2c^2 + 25ab^2c - 15b^4.$$

Multiplying this by 5, and adding on 3 times the square of the Parabolic Protomorph, we have

$$a(5a^2e - 35abd + 7ac^2 + 35b^2c),$$

where the expression in brackets is Prof. Sylvester's Protomorph of weight 4.

6. The case of $\kappa = 6$ deserves special attention.

In it $$a^2 = (A, B, C, D, E, F, G \between 1, t)^6,$$

when treated by the process of the preceding article, yields the pure reciprocant

$$a^5h - 15a^4bg - 21a^4cf - 21a^4de + 105a^3b^2f + 231a^3bce + 105a^3bd^2 + 105a^3c^2d$$
$$-420a^2b^3e - 1050a^2b^2cd - 280a^2bc^3 + 945ab^4d + 1260ab^3c^2 - 945b^5c,$$

which, since the complete primitive of the differential equation (obtained by equating this to zero) is given by (3) in the form

$$\left.\begin{aligned} x &= \int \frac{dt}{\sqrt{(A, B, \dots G \between 1, t)^6}} + \text{const.} \\ y &= \int \frac{tdt}{\sqrt{(A, B, \dots G \between 1, t)^6}} + \text{const.} \end{aligned}\right\},$$

may be called the Hyper-Elliptic Pure Reciprocant.

When the sextic function of t has two equal roots; *i.e.*, when

$$\frac{a^2}{(t+F)^2} = (A, B, C, D, E \between 1, t)^4,$$

the reasoning of Art. 4 shows that the form of this relation is permanent when we substitute for t and a their values in terms of τ and α, given by equations (14) and (15). Hence, if we eliminate the constants A, B, C, D, E, F, we shall arrive at the Pure Reciprocant whose complete primitive is

$$\left.\begin{aligned} x &= \int \frac{dt}{(t+F)\sqrt{(A, B, C, D, E \between 1, t)^4}} + \text{const.} \\ y &= \int \frac{t\,dt}{(t+F)\sqrt{(A, B, C, D, E \between 1, t)^4}} + \text{const.} \end{aligned}\right\},$$

where the second integral may be replaced by

$$y + Fx = \int \frac{dt}{\sqrt{(A, B, C, D, E \between 1, t)^4}} + \text{const.}$$

All the constants except F may be eliminated by the process of the preceding article; which $\left(\text{since } a = \frac{dt}{dx} \text{ so that } \frac{1}{a}\cdot\frac{d}{dx} = \frac{d}{dt}\right)$ is

equivalent to continued differentiation with respect to t, and gives

$$\frac{d^5}{dt^5}\left\{\frac{a^2}{(t+F)^2}\right\}=0.$$

After performing the differentiation and multiplying by $(t+F)^7$, to clear the equation of fractions, we shall obtain the following quintic in $t+F$,

$$\left\{(t+F)^5\frac{d^5}{dt^5}-10\,(t+F)^4\frac{d^4}{dt^4}+60\,(t+F)^3\frac{d^3}{dt^3}\right.$$
$$\left.-240\,(t+F)^2\frac{d^2}{dt^2}+600\,(t+F)\frac{d}{dt}-720\right\}a^2=0.$$

A final differentiation will give another quintic in $t+F$, and the resultant of these two quintics will be the Pure Reciprocant in question. Its value, expressed in terms of $a, b, c, \ldots$, appears to be too complicated to be of any use, and for this reason has not been calculated.

When the sextic in t has five equal roots, we may write

$$a^2=A\,(t+B)^5\,(t+C),$$

whence, by logarithmic differentiation,

$$\frac{2b}{a^2}=\frac{5}{t+B}+\frac{1}{t+C}.$$

Differentiating again with respect to x, and dividing by a,

$$\frac{2ac-4b^2}{a^4}=-\frac{5}{(t+B)^2}-\frac{1}{(t+C)^2}=-\frac{5}{(t+B)^2}-\left(\frac{2b}{a^2}-\frac{5}{t+B}\right)^2,$$

which reduces to $c\,(t+B)^2-10ab\,(t+B)+15a^3=0$ .... (17).

(The close resemblance of this to (4) may be noticed *en passant.*)

A final differentiation gives

$$d\,(t+B)^2-2\,(4ac+5b^2)\,(t+B)+35a^2b=0,$$

and the pure reciprocant we are in search of is obtained by eliminating B between this and (17); or, what is the same thing, it is the resultant of the two binary quadrics

$$(c,\ 5ab,\ 15a^3\between X,\ Y)^2,$$

$$(d,\ 4ac+5b^2,\ 35a^2b\between X,\ Y)^2.$$

The discriminant of the first of these is $5a^2\,(3ac-5b^2)$,

that of the second $35a^2bd-16a^2c^2-40ab^2c-25b^4$,

and their connective is $5a(3a^2d-abc-10b^3)$.

Hence, rejecting the factor $5a^2$ from their resultant, we obtain

$$5(3a^2d-abc-10b^3)^2-4(3ac-5b^2)(35a^2bd-16a^2c^2-40ab^2c-25b^4),$$

which divides again by a, and gives

$$45a^3d^2-450a^2bcd+192a^2c^3+400ab^3d+165ab^2c^2-400b^4c,$$

or the "Quasi-Discriminant" whose evanescence serves to mark points of closest contact of a cubical parabola with any curve.

Equation (17) is the first integral of the differential equation to the general cubical parabola whose coordinates are

$$\left.\begin{aligned} x &= \int \frac{dt}{\sqrt{A(t+B)^5(t+C)}} + \text{const.} \\ y &= \int \frac{t\,dt}{\sqrt{A(t+B)^5(t+C)}} + \text{const.} \end{aligned}\right\}.$$

If, now, we differentiate (17) with respect to B, and eliminate B from it by means of the resulting equation; we see that the discriminant of (17) regarded as a quadric in B, or $3ac-5b^2=0$, is a singular first integral of the differential equation to the cubical parabola. The geometrical property indicated is, that at some points at least on any curve where the Quasi-Discriminant vanishes it is possible to draw a common parabola through six consecutive points of the curve.

7. The present seems to be a fitting opportunity for pointing out the form of algebraic relation that must subsist between a and t in order that the differential equation, freed from arbitrary constants, of the curve implied by this relation may be expressed by the evanescence of a reciprocant.

The reasoning employed in Art. 4 will show that the most general algebraic relation of this kind is

$$a^m(1,\,t)^n+a^{m-1}(1,\,t)^{n+3}+a^{m-2}(1,\,t)^{n+6}+\ldots\ldots=0\ldots\ldots\;(18),$$

and that the final differential equation obtained from it will be of the form *Pure* Reciprocant $=0$;

provided only that the coefficients of all the quantics in t, which multiply the different powers of a, are either general or else connected by some *invariantive* condition; *e.g.*, $(1,\,t)^n$ may have two or more equal roots, and then its coefficients will be connected by an invariantive relation.

The second differential equation of any algebraic curve may, of course, be exhibited as an algebraic relation between a and t by eliminating x and y between the equation to the curve and the two other equations found by differentiating it twice with respect to x.

But (18) includes transcendental as well as algebraic curves.

As an easy example, the second differential equation of the conic

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0,$$

expressed in this form, is $\qquad a^2\Delta = (A+2Ht+Bt^2)^3,$

where Δ is the discriminant.

When the curve is unicursal, let u, v, w denote rational integral functions of θ, then

$$x=\frac{u}{w}, \quad \text{and} \quad dx=\frac{u'w-uw'}{w^2}\,.\,d\theta,$$

$$y=\frac{v}{w}, \quad \text{and} \quad dy=\frac{v'w-vw'}{w^2}\,.\,d\theta;$$

whence

$$t=\frac{v'w-vw'}{u'w-uw'} \quad \text{.........................} \quad (19),$$

and

$$a=\frac{dt}{dx}=\frac{w^2}{u'w-uw'}\cdot\frac{dt}{d\theta},$$

or, after some easy reductions,

$$a=\frac{w^3}{(u'w-uw')^3}\begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix} \text{..............} \quad (20),$$

and the elimination of θ between (19) and (20) gives the second differential equation of the curve in the form of an algebraic relation connecting a and t.

Thursday, February 11*th*, 1886.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Prof. P. H. Schoute, Ph.D., Professor of Mathematics at the Government University of Groningen, Netherlands, was elected a Member.

The following communications were made:—

On Perpetuant Reciprocants: Captain MacMahon, R.A.

Note on the Functions $Z(u)$, $\Theta(u)$, $\Pi(u, a)$: the President.

Note on a Z (u) Function: J. Griffiths, M.A.

On Polygons Circumscribed about a Conic and Inscribed in a Cubic: R. A. Roberts, M.A.

The following presents were received:—

Carte-de-Visite likeness of Mr. J. D. H. Dickson.

"Educational Times," for February.

"Journal of the Institute of Actuaries," Vol. xxv., Part III., CXXXVII., April, 1885.

"Proceedings of the Physical Society of London," Vol. VII., Part III., January, 1886.

"Bulletin des Sciences Mathématiques," for January and February, 1886.

"Annales de l'Ecole Polytechnique de Delft," Tome I., L. 3 and 4; Leide, 1885.

"Acta Mathematica," VII., 3; Stockholm, 1885.

"Atti della R. Accademia dei Lincei," Rendiconti, Vol. I., F. 28, 1885; Vol. II. F. 1; Roma, 1886.

"Beiblätter zu den Annalen der Physik und Chemie," B. x., St. 1; Leipzig, 1886.

"Sur le mouvement d'un corps pesant de révolution fixé par un point de son axe," par M. G. Darboux (from Journal de Mathématiques pures et appliquées).

"Appendix to Mathematical Questions and Solutions from Educational Times," Vol. XLIII.—"Solutions of some Old Questions," by Asûtosh Mukhopâdhyây, M.A., from the Author.

Perpetuant Reciprocants. By Captain MacMahon.

[*Read February 11th,* 1886.]

Reference is made to Prof. Sylvester's account of his discovery of Reciprocants, in *Nature* for January 7th, 1886; to several short articles on the same in recent numbers of the *Comptes Rendus*, and of the *Messenger of Mathematics.*

What is done, in this paper, is merely to present the numerical enumeration of the perpetuant reciprocants of the first six degrees, which is carried out on the same plan as that initiated by the author of their being for the allied Theory of Invariants, in Vol. v. of the *American Journal of Mathematics.*

The Theory of Invariants is, for the algebraist, concerned with the solutions of the lineo-linear partial differential equation

$$\lambda a\delta_b+\mu b\delta_c+\nu c\delta_d+\ldots=0;$$

these are now termed *binariants;* and, were we to calculate any such general form, we would find that the coefficient of every term was partly numerical and partly composed of the letters λ, μ, ν, ..., and that, on putting $\lambda=\mu=\nu=\ldots=1$, the binariant would become a

non-unitary symmetric function (*i.e.*, one involving no root to unit power) of the general equation of the n^{th} degree affected simply with literal coefficients. Any binariant which cannot be expressed as a function of other binariants of lower degree and weight, the number of letters which we are allowed to employ being infinite, may in a certain sense be termed a perpetuant; but, fixing upon certain of such forms of a given degree and weight, it will often happen that the remaining forms are expressible in terms of them and of others of lower degree and weight, and these, as not being linearly independent of the forms chosen, are not enumerated; the forms which are counted, not being linearly connected with lower forms, are termed by writers, both here and on the Continent, "asyzygetic"; where such a linear relation does exist, it is termed a "syzygy." Some confusion of terms has lately arisen as to the meaning of the word "syzygant." When the term was first employed, it was used to denote the left-hand side of a syzygy; thus, suppose $V=0$ was the syzygy, then V was the syzygant; but latterly another meaning has crept in, as it was found an extremely convenient word for expressing another idea, whereas in its former signification it was slightly redundant. An irreducible form of degree κ which becomes reducible when multiplied by a^{λ} is said to be a $(\kappa+\lambda)^{ic}$ syzygant.

Thus $$(a^2d-3abc+2b^3)^2+4(ac-b^2)^3-a^2\Delta = 0$$

is a syzygy, Δ denoting the discriminant of the cubic; Δ is called a sextic syzygant.

The perpetuant theory of invariants, that is, the enumeration of the ground forms of the quantic of infinite order, is complete (*vide* Vol. VI., *Amer. Jour. of Math.*); this resulted from the employment of the partition symbols for symmetric functions, an idea due to Hirsch at the beginning of this century. No symbolical method of treating reciprocants has yet been found, and accordingly the discussion here is only carried on as far as forms which involve products of six differential coefficients.

Representing as usual the 2nd, 3rd, 4th, &c. ... differential coefficients of y with respect to x (x and y being Cartesian coordinates) by the letters $a, b, c, \ldots$ respectively, a reciprocant satisfies the partial differential equation

$$3a^2\delta_b+10ab\delta_c+(15ac+10b^2)\,\delta_d+\ldots = 0,$$

and, considering the letters $a, b, c, \ldots$ to be of weights $0, 1, 2, \ldots,$ the number of asyzygetic solutions of a given degree-weight (θ, w) in the letters is equal to the excess of the number of partitions of w into

θ parts or fewer over the number of partitions of $w-1$ into $\theta+1$ parts or fewer; that is,

$$\text{Co. } z^{\theta}x^{w} \frac{1}{(1-z)(1-zx)(1-zx^2)(1-zx^3)\ldots},$$

$$-\text{Co. } z^{\theta+1}x^{w-1} \frac{1}{(1-z)(1-zx)(1-zx^2)(1-zx^3)\ldots},$$

or it is
$$\text{Co. } x^{w} \frac{1-x-x^{\theta+1}}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^{\theta+1})}.$$

This generating function is thus seen to depend upon an observation of the partitions of the numbers w and $w-1$, as was the case in the theory of invariants; this follows naturally directly we agree to consider $a, b, c, \ldots$ of the weights $0, 1, 2, \ldots$, and, for so doing, there is solid reason in the theory of invariants, as may be seen at once from the connexion of the letters with the roots of the quantic under discussion; in the theory of reciprocants, however, there do not appear to be grounds for absolutely fixing the weights of the letters in the same manner, and for any particular purpose we may suppose their weights to be $n, n+1, n+2, \ldots$ *Mutatis mutandis*, correct theorems will naturally be obtained on this hypothesis. Suppose for a moment we consider $n=1$, or the letters $a, b, c, \ldots$ to be of weights $1, 2, 3, \ldots$; an inspection of the quadro-linear reciprocant annihilator shows that, when the latter operates upon any product of letters, there is no diminution in the weight of the resultant terms, an observation which points to the conclusion that an examination of the partitions of a single weight is alone necessary for the determination of the number of asyzygetic reciprocants of degree-weight (θ, w), where the number $w+\theta$ is fixed; w being here employed in its ordinary sense.

For, suppose a term of degree θ,

$$a^{\alpha}b^{\beta}c^{\gamma};$$

when $n=0$, its weight will be

$$\beta+2\gamma,$$

whilst, when $n=1$, what I will call its hyper-weight is

$$\alpha+2\beta+3\gamma,$$

or
$$\text{hyper-weight} - \text{weight} = \theta.$$

It follows that, for a given hyper-weight, the number of reciprocants (θ, w) is the excess of the number of partitions of the hyper-

weight ($=w+\theta$) into exactly θ parts, over the number of its partitions into exactly $\theta+1$ parts.

Analytically, this is

$$(\text{Co.}\, z^{\theta}x^{w+\theta}-\text{Co.}\, z^{\theta+1}x^{w+\theta})\,\frac{1}{1-zx\,.\,1-zx^2\,.\,1-zx^3\,.\,1-zx^4\,\ldots}$$

$$=\text{Co.}\, z^{\theta}x^{w+\theta}\,\frac{\left(1-\frac{1}{z}\right)}{1-zx\,.\,1-zx^2\,.\,1-zx^3\,.\,1-zx^4\,\ldots}$$

$$=\text{Co.}\, x^{w+\theta}\left\{-\frac{1}{1-x\,.\,1-x^2\ldots 1-x^{\theta+1}}+\frac{2}{1-x\,.\,1-x^2\ldots 1-x^{\theta}}-\frac{1}{1-x\,.\,1-x^2\ldots 1-x^{\theta-1}}\right\}$$

$$=\text{Co.}\, x^{w}\,\frac{1-x-x^{\theta+1}}{1-x\,.\,1-x^2\ldots 1-x^{\theta+1}},$$

the same expression before obtained.

Ex. gr., write down the partitions of the number 7, according to number of parts, as follows:—

Col. =	1	2	3	4	5	6	7
	7	61	51^2	41^3	31^4	21^5	1^7
		52	421	321^2	2^21^3		
		43	3^21	2^31			
			32^2				
No. =	1	3	4	3	2	1	1

whence, for a hyper-weight $=7$, we have asyzygetic forms (θ, w),

$$(3, 4),$$
$$(4, 3),$$
$$(5, 2),$$
$$(7, 0),$$

viz., these are $\quad 5a^2e-35abd+7ac^2+35b^2c,$

$$a\,(9a^2d-45abc+40b^3),$$
$$a^3\,(3ac-5b^2),$$
$$a^7.$$

It may here be remarked that we in reality need only to consider the non-unitary partitions of the hyper-weight, for the partitions of a number into exactly $\theta+1$ parts may be derived from those containing exactly θ parts as follows :—

(1) Transform each partition, whose highest part is not repeated, by diminishing the highest part by unity and adding a part unity.

(2) Add the non-unitary partitions containing exactly $\theta+1$ parts.

Thus clearly the number of asyzygetic reciprocants will be the number of partitions into exactly θ parts, in which the first part is repeated, diminished by the number of non-unitary partitions composed of exactly $\theta+1$ parts; or, what is the same thing, the number of non-unitary partitions, containing θ as a highest part, diminished by the number of non-unitary partitions containing exactly $\theta+1$ parts.

Ex. gr., for degree-weight (4, 12), we have

$$\begin{array}{rcc} & 4^3 & 42^4 \\ & 4^2 2^2 & 3^2 2^3 \\ & 43^2 2 & \\ & 42^4 & \\ \cline{2-3} \text{No.} = & 4 & 2 \end{array}$$

whence $4-2=2$, asyzygetic reciprocants; these are, in fact, the forms whose leading terms are $a^3 i$, and $a^2 cg$.

This method, based on the consideration of the hyper-weight $w+\theta$, as only necessitating the tabulation of the partitions of a single weight, is not without considerable practical advantage, in which it is needless to say the theory of invariants does not participate.

(2) Recalling that the number of asyzygetic reciprocants of degree-weight (θ, w) is the coefficient of x^w in

$$\frac{1-x-x^{\theta+1}}{1-x\,.\,1-x^2\,\ldots\,1-x^{\theta+1}},$$

we say that the generating function (G. F.) for such is

$$\frac{1-x-x^{\theta+1}}{(1)(2)\,\ldots\,(\theta+1)},$$

wherein as usual (μ) denotes $1-x^\mu$ for brevity.

For degree 0, G. F. is

$$\frac{1-2x}{1-x} \quad \text{or} \quad 1-\frac{x}{1-x},$$

indicating the numerical constant unity of degree-weight (0, 0).

For degree 1, G. F. is $$\frac{1-x-x^2}{(1)(2)},$$

which is $$1-\frac{x^3}{(1)(2)},$$

showing that there is a form degree-weight (1, 0), viz.,

$$a.$$

For degree 2,

$$\text{G. F.} = \frac{1-x-x^3}{(1)(2)(3)} = 1+x^2-\frac{x^5}{(1)(2)}-\frac{x^7}{(1)(2)(3)},$$

so that $$\text{G. F.} = 1+x^2,$$

from which, subtracting the square of the form of degree-weight (1, 0), there remains $$x^2,$$

as the G. F. for quadric perpetuants; this is

$$(2,\ 2) = 3ac-5b^2.$$

For degree 3, $$\text{G. F.} = \frac{1-x-x^4}{(1)(2)(3)(4)}$$

$$= 1+x^2+x^3+x^4+x^6-\frac{x^7+x^{11}}{(1)(2)(3)}-\frac{x^9}{(1)(2)(4)}-\frac{x^{16}}{(1)(2)(3)(4)},$$

so that $$\text{G. F.} = 1+x^2+x^3+x^4+x^6.$$

This includes those of degree 2, each multiplied by a, so that, subtracting the total G. F. of forms of degree 2, there remains

$$x^3+x^4+x^6,$$

as the G. F. for cubic perpetuants.

These are

$(3\,.\,3) = 9a^2d-45abc+40b^3,$

$(3\,.\,4) = 5a^2e-35abd+7ac^2+35b^2c,$

$(3\,.\,6) = a^2g-12abf-450ace+588ad^2+792b^2e-2772bcd+1925c^3,$

forms already calculated by Prof. Sylvester.

For degree 4, $$\text{G. F.} = \frac{1-x-x^5}{(1)(2)(3)(4)(5)},$$

to expand which we have

Ind. x	$1 \div (1)(2)(3)(4)(5)$	$-x$	$-x^5$	$=$
0	1			+1
1	1	−1		0
2	2	−1		+1
3	3	−2		+1
4	5	−3		+2
5	7	−5	−1	+1
6	10	−7	−1	+2
7	13	−10	−2	+1
8	18	−13	−3	+2
9	23	−18	−5	0
10	30	−23	−7	0
11	37	−30	−10	−3
12	47	−37	−13	−3
13	57	−47	−18	−8
14	70	−57	−23	−10
15	84	−70	−30	−16
⋮	⋮	⋮	⋮	⋮

from which there is no practical doubt that the highest power of x with a positive coefficient is 8; that such is actually the case is proved by the identity :—

$$\frac{1-x-x^5}{(1)(2)(3)(4)(5)} = 1+x^2+x^3+2x^4+x^5+2x^6+x^7+2x^8$$

$$-\frac{x^{11}(1+x^2+x^4)}{(1)(4)} - \frac{2x^{11}}{(1)(2)(3)(4)} - \frac{x^{20}}{(1)(4)(5)} - \frac{2x^{13}+x^{21}}{(1)(3)(4)(5)}.$$

We therefore have

$$\text{G. F.} = 1+x^2+x^3+2x^4+x^5+2x^6+x^7+2x^8,$$

and, subtracting from it

$$1+x^2+x^3+x^4+x^6$$

due to the forms of degree 3, and x^4

due to the square of the quadric perpetuant, we find that

G. F. for quartic perpetuants is $x^5+x^6+x^7+2x^8$.

These are, as far as weight 8,

$$(4.5) = 15a^3f-140a^2be-63a^2cd+455ab^2d+140abc^2-455b^3c,$$

$$(4.6) = 240a^2ce-400ab^2e-315a^2d^2+1470abcd-1008ac^3-35b^2c^2,$$

$$(4.7) = a^3h-15a^2bg-462a^2cf+726a^2de+840ab^2f+612abce$$
$$-5124abd^2+3003ac^2d-3960b^3e+13860b^2cd-9625bc^3,$$

$$(4.8)_1 = 3a^3i-55a^2bh-1752a^2cg+4950a^2df-2046a^2e^2+3250ab^2g$$
$$-4056abcf-18018abde+25935a^2c^2e-11466acd^2$$
$$-13260b^3f+27846b^2ce+37128b^2d^2-51051bc^2d,$$

$$(4.8)_2 = 63a^2cg-693a^2df+704a^2e^2-105ab^2g+2709bcf-3388abde$$
$$-6790a^2c^2e+7938acd^2-1820b^3f+9366b^2ce+5096b^2d^2$$
$$-26411bc^2d+13475c^4.$$

For degree 5, $\quad$ G. F. $= \dfrac{1-x-x^6}{(1)(2)(3)(4)(5)(6)}$;

and the following identity may be verified, viz.,

$$\frac{1-x-x^6}{(1)(2)(3)(4)(5)(6)} = 1+x^2+x^3+2x^4+2x^5+3x^6$$
$$+2x^7+4x^8+3x^9+4x^{10}+2x^{11}+3x^{12}$$
$$-\frac{x^{13}+x^{17}}{(1)(2)(3)}-\frac{x^{15}+x^{18}}{(1)(2)(4)}-\frac{3(x^{15}+x^{24})}{(1)(2)(3)(4)}$$
$$-\frac{2(x^{16}+x^{17}+x^{19})}{(1)(4)(5)(6)}-\frac{x^{23}+x^{26}}{(1)(2)(3)(4)(6)}-\frac{x^{25}}{(1)(2)(4)(5)(6)}$$
$$-\frac{3(x^{25}+x^{33})}{(1)(3)(4)(5)(6)}-\frac{x^{32}}{(1)(2)(3)(4)(5)(6)}.$$

For as before

Ind. x	$1 \div (1)(2)(3)(4)(5)(6)$	$-x$	$-x^6$	G. F. =
0	1			1
1	1	1		0
2	2	1		1
3	3	2		1
4	5	3		2
5	7	5		2
6	11	7	1	3
7	14	11	1	2
8	20	14	2	4
9	26	20	3	3
10	35	26	5	4
11	44	35	7	2
12	58	44	11	3
13	71	58	14	-1
14	90	71	20	-1
15	110	90	26	-6
16	136	110	35	-9
17	163	136	44	-17
⋮	⋮	⋮	⋮	⋮

So that

$$\text{G. F.} = 1+x^2+x^3+2x^4+2x^5+3x^6+2x^7+4x^8+3x^9+4x^{10}+2x^{11}+3x^{12},$$

subtracting from which the total G. F. for degree 4, viz.,

$$1+x^2+x^3+2x^4+x^5+2x^6+x^7+2x^8,$$

and further that for the product of the quadric perpetuant with each of the cubic perpetuants, corresponding to the non-unitary partition (32) of the degree 5, viz.,

$$x^2(x^3+x^4+x^6) = x^5+x^6+x^8,$$

so that the total subtrahend is

$$1+x^2+x^3+2x^4+2x^5+3x^6+x^7+3x^8,$$

there remains the G. F. for quintic perpetuants, viz.,

$$x^7+x^8+3x^9+4x^{10}+2x^{11}+3x^{12};$$

of which

$$(5\,.\,7)=120a^3cf-200a^2b^2f-195a^3de-145a^2bce+1000ab^3e+1365a^2bd^2$$
$$-777a^2c^2d-3710ab^2cd+2485abc^3+105b^3c^2,$$

$$(5\,.\,8)=8757a^3df-10771a^3e^2-43785a^2bcf+69062a^2bde+100310a^2c^2e$$
$$-128331a^2cd^2+38920ab^3f-177030ab^2ce-109564ab^2d^2$$
$$+478975abc^2d-229075ac^4-840b^4e+2940b^3cd.$$

Passing to degree 6, we have, for the expansion of

$$\frac{1-x-x^7}{(1)(2)(3)(4)(5)(6)(7)},$$

Ind. x	$1\div(1)(2)(3)(4)(5)(6)(7)$	$-x$	$-x^7$	G. F. =
0	1			+ 1
1	1	− 1		0
2	2	− 1		+ 1
3	3	− 2		+ 1
4	5	− 3		+ 2
5	7	− 5		+ 2
6	11	− 7		+ 4
7	15	− 11	− 1	+ 3
8	21	− 15	− 1	+ 5
9	28	− 21	− 2	+ 5
10	38	− 28	− 3	+ 7
11	49	− 38	− 5	+ 4
12	65	− 49	− 7	+ 9
13	82	− 65	−11	+ 6
14	105	− 82	−15	+ 8
15	131	−105	−21	+ 5
16	164	−131	−28	+ 5
17	201	−164	−38	− 1
18	248	−201	−49	− 2
19	300	−248	−65	−13
⋮	⋮	⋮	⋮	⋮

and from previous experience we may assume that there exist no more positive terms, so that

$$\text{G. F.} = 1+x^2+x^3+2x^4+2x^5+4x^6+3x^7+5x^8+5x^9+7x^{10}+4x^{11}$$
$$+9x^{12}+6x^{13}+8x^{14}+5x^{15}+5x^{16}.$$

Subtracting from this the total G. F. for degree 5, we get

$$x^6+x^7+x^8+2x^9+3x^{10}+2x^{11}+6x^{12}+6x^{13}+8x^{14}+5x^{15}+5x^{16}.$$

Due to the partition (42), there is a further subtrahend

$$x^2\,(x^5+x^6+x^7+2x^8),$$

and a subtrahend due to (33), viz.,

$$x^6+x^7+x^8+x^9+x^{10}+x^{12},$$

while that due to (222) is obviously

$$x^6,$$

making a total subtrahend due to the non-unitary partitions of 6 of

$$2x^6+2x^7+2x^8+2x^9+3x^{10}+x^{12};$$

subtracting this, therefore, we find

$$\text{G. F.} = -x^6-x^7-x^8+2x^{11}+5x^{12}+6x^{13}+8x^{14}+5x^{15}+5x^{16},$$

wherein the negative terms indicate that the compound reciprocants we have subtracted are not all linearly independent, but that certain syzygies exist between them, which must be discovered, and their G. F. added to the above expression, according to Prof. Sylvester's theory as corrected by Hammond, which establishes that, if

$A =$	No. of	asyzygetic forms	of weight	w,
$C =$	,,	compound	,,	,,
$S =$	,,	syzygies	,,	,,
$P =$	,,	perpetuants	,,	,,

then $$P = A-C+S.$$

That portion of a reciprocant not containing a, or its residue (*cf.* Sylvester, *Amer. Jour. of Math.*, Vol. v.), satisfies the partial differential equation arrived at by putting $a = 0$ in the reciprocant annihilator;

that is, the equation

$$10b^2\delta_d + 35bc\delta_e + (56bd + 35c^2)\,\delta_f + \ldots = 0,$$

an examination of which shows—

(1) That a residue may consist wholly of the letters b and c.

(2) That if it contains any higher letters it must contain at least two such.

The reciprocant and residue annihilators are not transformable one into the other, indicating that the residue is not, in general, itself a reciprocant; the theory thus presents an important difference from that of binariants, in that the binariant operator

$$\lambda a\delta_b + \mu b\delta_c + \nu c\delta_d + \ldots$$

is at once convertible to the residue operator

$$\mu b\delta_c + \nu c\delta_d + \ldots,$$

showing, what is well known, that a binariant residue is itself a binariant of another system of elements; this principle, which has led to so great an advancement of the theory of invariants, is consequently inapplicable here.

To every identity between reciprocant residues obviously corresponds a reciprocant syzygy, and, failing any other method, these must be determined by simple observation.

Noting that the G. F. for sextic compounds, arising from the non-unitary partitions of 6, is

$$2x^6 + 2x^7 + 2x^8 + 2x^9 + 3x^{10} + x^{12},$$

it is remarked that the possible syzygies are five in number, viz., one each of the weights 6, 7, 8, 9, 10;

there cannot be more than one syzygy of weight 10, since there is no syzygy of lower degree of weight 8.

Representing the residue of a form (θ, w) by $R(\theta, w)$, we find

$$64R\,(2, 2)^3 + 5R\,(3, 3)^2 = 0,$$

whence a syzygy of weight 6, (G. F. $= x^6$); also

$$13R\,(3, 3)\,R\,(3, 4) - 8R\,(2, 2)\,R\,(4, 5) = 0,$$

or a syzygy of weight 7, (G. F. $= x^7$).

Again, $$7R(2, 2)\,R(4, 6) - R(3, 4)^2 = 0,$$

$$5R(3, 3)\,R(3, 6) - 8R(2, 2)\,R(4, 7) = 0,$$

indicating syzygies of weights 8 and 9, (G. F. $= x^8 + x^9$).

To show the syzygy of weight 10, observe that from the quartic perpetuants of weight 8, we have

$$7R(4, 8)_1 - 51R(4, 8)_2 = -357\,(792b^3ce - 2772bc^2d + 1925c^4),$$

and $$R(3, 4)\,R(3, 6) = 35b^2c\,(792b^2e - 2772bcd + 1925c^3),$$

whence

$$51R(3, 4)\,R(3, 6) - 7R(2, 2)\,R(4, 8)_1 + 51R(2, 2)\,R(4, 8)_2 = 0,$$

$$(\text{G. F.} = x^{10});$$

there can be no more, and the complete G. F. for sextic syzygies is

$$x^6 + x^7 + x^8 + x^9 + x^{10},$$

adding which to the foregoing G. F., it is found that G. F. for sextic perpetuants is

$$x^9 + x^{10} + 2x^{11} + 5x^{12} + 6x^{13} + 8x^{14} + 5x^{15} + 5x^{16}.$$

Until more forms have been calculated, or a new principle discovered, it is not practicable to proceed to another degree.

So far, the G. F. for perpetuants (θ, w) is the coefficient of $z^\theta x$ in

$$1 + z + z^2x^2 + z^3(x^3 + x^4 + x^6) + z^4(x^5 + x^6 + x^7 + 2x^8)$$
$$+ z^5(x^7 + x^8 + 3x^9 + 4x^{10} + 2x^{11} + 3x^{12})$$
$$+ z^6(x^9 + x^{10} + 2x^{11} + 5x^{12} + 6x^{13} + 8x^{14} + 5x^{15} + 5x^{16}) + \dots,$$

of which I have given the values up to weight 8 inclusive.

Note on the Functions $Z(u)$, $\Theta(u)$, $\Pi(u, a)$.

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§ 1. The present note relates to the three fundamental formulæ

$$\text{(i.)}\quad \mathrm{Z}(u)+\mathrm{Z}(v)-\mathrm{Z}(u+v) = k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v),$$

$$\text{(ii.)}\quad \frac{\Theta(u+a)\,\Theta(u-a)\,\Theta^2(0)}{\Theta^2(u)\,\Theta^2(a)} = 1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a,$$

$$\text{(iii.)}\quad \Pi(u, a) = a\mathrm{Z}(u)+\tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}.$$

The first of these formulæ is in effect Legendre's addition-equation for the second elliptic integral. The second and third were given in the *Fundamenta Nova*, the second being deduced from the third in § 53, and the third being established in §§ 51, 52, by means of q-series. In § 53, Jacobi deduces the first formula from the third, but he had previously given, in § 49, an independent proof of the first formula by Elliptic Functions, the notation employed being however Legendrian, in order no doubt that the result might be obtained in Legendre's form.

The object of this note is to show how extremely simply the three formulæ may be established independently of each other by elementary Elliptic Functions, and to point out the close connexion existing between (i.) and (ii.).

§ 2. The definitions of the functions $\mathrm{Z}(u)$, $\Theta(u)$, $\Pi(u, a)$ are taken to be

$$\mathrm{Z}(u) = \int_0^u \operatorname{dn}^2 u\,du - \frac{E}{K}\,u,$$

$$\Theta(u) = e^{\int_0^u \mathrm{Z}(u)\,du},$$

$$\Pi(u, a) = \int^u \frac{k^2 \operatorname{sn}^2 u \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a\,du}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a};$$

and I take as starting point the fundamental formula

$$\begin{aligned} & k^2 \operatorname{sn}^2(u-a) - k^2 \operatorname{sn}^2(u+a) \\ &= k^2 \operatorname{cn}^2(u+a) - k^2 \operatorname{cn}^2(u-a) \\ &= \quad \operatorname{dn}^2(u+a) - \operatorname{dn}^2(u-a) \\ &= -\frac{4k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}{(1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a)^2} \quad \text{..................(A)}, \end{aligned}$$

which is derivable at sight from the ordinary addition formulæ, giving the sn, cn or dn of $u \pm a$ in terms of the sn's, cn's, and dn's of u and a.

§ 3. Integrating (A) with respect to a, we find

$$Z(u+a) + Z(u-a) = C - \frac{1}{\operatorname{sn}^2 u} \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} \quad \text{......}(A_1),$$

where C is the constant of integration, and is therefore independent of a.

1°. Putting $a = u$ in (A_1), we have

$$Z(2u) = C - \frac{1}{\operatorname{sn}^2 u} \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^4 u},$$

whence, by subtraction,

$$\begin{aligned} & Z(u+a) + Z(u-a) - Z(2u) \\ &= \frac{1}{\operatorname{sn}^2 u} \left\{ \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^4 u} - \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} \right\} \\ &= \operatorname{sn} 2u \frac{k^2 (\operatorname{sn}^2 u - \operatorname{sn}^2 a)}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} \\ &= k^2 \operatorname{sn}(u+a) \operatorname{sn}(u-a) \operatorname{sn} 2u \quad \text{..............................}(\alpha). \end{aligned}$$

2°. Putting $a = 0$ in (A_1), we have

$$2Z(u) = C - \frac{1}{\operatorname{sn}^2 u} 2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u,$$

whence, by subtraction,

$$Z(u+a) + Z(u-a) - 2Z(u) = -\frac{2k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn}^2 a}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} \quad \text{......}(\beta).$$

The equation (α) is equivalent to the addition-equation (i.), for, on re-

placing $u+a$ and $u-a$ by u and v, it becomes

$$\mathrm{Z}(u)+\mathrm{Z}(v)-\mathrm{Z}(u+v)=k^2\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{sn}\,(u+v).$$

By integrating the equation (β) with respect to u between the limits u and 0, we find

$$\int_0^u \mathrm{Z}(u+a)\,du+\int_0^u \mathrm{Z}(u-a)\,du-2\int_0^u \mathrm{Z}(u)\,du=\log(1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 a).$$

Now
$$\int_0^u \mathrm{Z}(u+a)\,du=\log\frac{\Theta(u+a)}{\Theta(a)},$$
$$\int_0^u \mathrm{Z}(u-a)\,du=\log\frac{\Theta(u-a)}{\Theta(a)}.$$

The equation just obtained may therefore be written

$$\log\frac{\Theta(u+a)}{\Theta(a)}+\log\frac{\Theta(u-a)}{\Theta(a)}-2\log\frac{\Theta(u)}{\Theta(0)}=\log(1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 a),$$

that is,
$$\frac{\Theta(u+a)\,\Theta(u-a)\,\Theta^2(0)}{\Theta^2(u)\,\Theta^2(a)}=1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 a\ldots\ldots\ldots\ldots(\beta_1),$$

which is the formula (ii.).

By integrating (β) with respect to a, instead of with respect to u, we find

$$\int_0^a \mathrm{Z}(u+a)\,da+\int_0^a \mathrm{Z}(u-a)\,da-2a\,\mathrm{Z}(u)$$
$$=-2\int_0^a \frac{k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u\,\mathrm{sn}^2 a\,da}{1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 a},$$

that is,
$$\log\frac{\Theta(a+u)}{\Theta(a-u)}-2a\mathrm{Z}(u)=-2\Pi(a,u),$$

or, on transposing u and a,

$$\tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}+a\mathrm{Z}(u)=\Pi(u,a)\ldots\ldots\ldots\ldots\ldots\ldots(\beta_2),$$

which is the formula (iii.).

Thus, starting with the identical equation (A), and integrating it with respect to a, we obtain the two forms (α) and (β) of the addition-equation; of these (α) is the same as (i.), and (β) gives rise to (ii.) by integration with respect to u, and to (iii.) by integration with

respect to a. The three formulæ (i.), (ii.), (iii.) have therefore been derived independently from the elementary identity (A).

§ 4. In the preceding investigation, the results (α) and (β) were obtained by integrating (A) with respect to a, and determining the constant of integration by putting $a = u$ to obtain (α), and by putting $a = 0$ to obtain (β). We may however, if we please, derive (α) from (β), instead of deducing it independently from (A) by a separate determination of the constant; for, putting $u = a$ in (β), we have

$$\mathrm{Z}(2u) - 2\mathrm{Z}(u) = -\frac{2k^2 \operatorname{sn}^3 u \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^4 u},$$

whence, by subtraction,

$$\begin{aligned}\mathrm{Z}(u+a) + \mathrm{Z}(u-a) - \mathrm{Z}(2u) &= \frac{2k^2 \operatorname{sn}^3 u \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^4 u} - \frac{2k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn}^2 a}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} \\ &= k^2 \operatorname{sn}(u+a) \operatorname{sn}(u-a) \operatorname{sn} 2u.\end{aligned}$$

If, therefore, starting with (A), we integrate it with respect to a, between the limits a and 0, thus obtaining (β), we may deduce therefrom all three formulæ (i.), (ii.), (iii.); viz., (i.) by putting $u = a$ and subtracting, (ii.) by integrating (β) with respect to u, and (iii.) by integrating (β) with respect to a.

§ 5. To deduce (ii.) from (i.), we may proceed as follows :—

Substituting $u+a$ and $u-a$ for u and v, (i.) becomes

$$\mathrm{Z}(u+a) + \mathrm{Z}(u-a) - \mathrm{Z}(2u) = k^2 \operatorname{sn}(u+a) \operatorname{sn}(u-a) \operatorname{sn} 2u;$$

whence, putting $a = 0$,

$$2\mathrm{Z}(u) - \mathrm{Z}(2u) = k^2 \operatorname{sn}^2 u \operatorname{sn} 2u,$$

and therefore, by subtraction,

$$\begin{aligned}\mathrm{Z}(u+a) + \mathrm{Z}(u-a) - 2\mathrm{Z}(u) &= k^2 \operatorname{sn} 2u \left\{\operatorname{sn}(u+a) \operatorname{sn}(u-a) - \operatorname{sn}^2 u\right\} \\ &= -\frac{2k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn}^2 a}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a},\end{aligned}$$

from which (ii.) follows at once by integration with respect to a, as in § 3.

But, by the following process we may derive (ii.) even more directly from (i.) by integrating (i.) while it still remains in the form of an addition-equation. Writing (i.) in the form

$$\mathrm{Z}(u+a) + \mathrm{Z}(u-a) - \mathrm{Z}(2u) = k^2 \operatorname{sn}(u+a) \operatorname{sn}(u-a) \operatorname{sn} 2u,$$

we notice that the right-hand member of this equation

$$= k^2 \,\mathrm{sn}\,(u+a)\,\mathrm{sn}\,(u-a)\,\mathrm{sn}\,\{(u+a)+(u-a)\}$$

$$= k^2 \frac{\mathrm{sn}^2(u+a)\,\mathrm{sn}\,(u-a)\,\mathrm{cn}\,(u-a)\,\mathrm{dn}\,(u-a) + \mathrm{sn}^2(u-a)\,\mathrm{sn}\,(u+a)\,\mathrm{cn}\,(u+a)\,\mathrm{dn}\,(u+a)}{1-k^2\,\mathrm{sn}^2(u+a)\,\mathrm{sn}^2(u-a)}$$

$$= -\tfrac{1}{2}\frac{d}{du}\log\{1-k^2\,\mathrm{sn}^2(u+a)\,\mathrm{sn}^2(u-a)\}.$$

Integrating therefore the equation, as it stands, with respect to u between the limits u and 0, we find

$$\frac{\Theta^2(u+a)\,\Theta^2(u-a)\,\Theta(0)}{\Theta^4(a)\,\Theta(2u)} = \frac{1-k^2\,\mathrm{sn}^4 a}{1-k^2\,\mathrm{sn}^2(u+a)\,\mathrm{sn}^2(u-a)}.$$

Putting $a=0$, this equation becomes

$$\frac{\Theta^4(u)\,\Theta(0)}{\Theta^4(0)\,\Theta(2u)} = \frac{1}{1-k^2\,\mathrm{sn}^4 u},$$

whence, by division,

$$\frac{\Theta^2(u+a)\,\Theta^2(u-a)\,\Theta^4(0)}{\Theta^4(u)\,\Theta^4(a)} = \frac{(1-k^2\,\mathrm{sn}^4 u)(1-k^2\,\mathrm{sn}^4 a)}{1-k^2\,\mathrm{sn}^2(u+a)\,\mathrm{sn}^2(u-a)}$$

$$= (1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 a)^2,$$

which is equivalent to the formula (ii.).

§ 6. So far as I know, the close connexion between the "addition-equation" (ii.) for the function Θ, and the addition-equation for the second elliptic integral, has not been specially remarked. In my lectures on Elliptic Functions, I have been in the habit of following Jacobi in proving (ii.) by means of the third elliptic integral. While working at formulæ connected with the Zeta Function, I recently noticed that (ii.) was derivable immediately from (i.) by integration, so that, in order to prove (ii.), it was unnecessary either to have recourse to the third elliptic integral or to use q-series for the Θ's; and this led me to remark how simply (i.), (ii.), and (iii.) may all be deduced from the identity (A), and how closely they are related. It will be observed that the method of proving (i.) in § 3 is practically the same as the method given by Jacobi, in Legendrian notation, in § 49 of the *Fundamenta Nova*.

§ 7. The quantity $k^2\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{sn}\,(u+v)$, which forms the right-hand member of the addition-equation

$$\mathrm{Z}(u)+\mathrm{Z}(v)-\mathrm{Z}(u+v) = k^2\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{sn}\,(u+v),$$

does not, when so expressed, appear to possess any interesting or remarkable property. If, however, we transform the arguments by replacing u and v by $u+a$ and $u+b$ (so that the letter u occurs in each argument), we see that it is an expression which possesses the remarkable property of being an exact differential coefficient with respect to u;

viz., it
$$= k^2 \operatorname{sn}(u+a) \operatorname{sn}(u+b) \operatorname{sn}(2u+a+b)$$
$$= \frac{k^2 s_1 s_2 (s_1 c_2 d_2 + s_2 c_1 d_1)}{1-k^2 s_1^2 s_2^2}$$
$$= -\tfrac{1}{2} \frac{d}{du} \log(1-k^2 s_1^2 s_2^2),$$

where s_1, c_1, d_1, s_2, c_2, d_2 denote the sn, cn, dn of $u+a$ and $u-a$ respectively.

§ 8. The arguments $u+a$ and $u+b$ are not more general than $u+a$ and $u-a$, but it may be worth while perhaps to notice the result obtained by integrating the addition-equation in the form

$$Z(u+a)+Z(u+b)-Z(2u+a+b) = k^2 \operatorname{sn}(u+a) \operatorname{sn}(u+b) \operatorname{sn}(2u+a+b)$$

with respect to u.

The limits being u and 0, we thus find

$$\frac{\Theta^2(u+a)\ \Theta^2(u+b)\ \Theta(a+b)}{\Theta^2(a)\ \Theta^2(b)\ \Theta(2u+a+b)} = \frac{1-k^2 \operatorname{sn}^2 a \operatorname{sn}^2 b}{1-k^2 \operatorname{sn}^2(u+a) \operatorname{sn}^2(u+b)},$$

from which (ii.) follows at once by putting $u = -a$ or $u = -b$.

Replacing $u+a$ and $u+b$ by x and y, and finally writing $-a$ instead of u, this equation becomes

$$\frac{\Theta^2(x)\ \Theta^2(y)\ \Theta(x+y+2a)}{\Theta^2(x+a)\ \Theta^2(y+a)\ \Theta(x+y)} = \frac{1-k^2 \operatorname{sn}^2(x+a) \operatorname{sn}^2(y+a)}{1-k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y},$$

which, making a slight further change of notation, may be written also in the form

$$\frac{\Theta^2(x+a)\ \Theta^2(y+a)\ \Theta(x+y-2a)}{\Theta^2(x-a)\ \Theta^2(y-a)\ \Theta(x+y+2a)} = \frac{1-k^2 \operatorname{sn}^2(x-a) \operatorname{sn}^2(y-a)}{1-k^2 \operatorname{sn}^2(x+a) \operatorname{sn}^2(y+a)}.$$

These two formulæ are therefore deducible by direct integration from the addition-equation for the second elliptic integral, subject only to changes in the letters by which the arguments are expressed.

On Polygons Circumscribed about a Conic and Inscribed in a Cubic.

By Mr. R. A. Roberts, M.A.

[*Read Feb.* 11*th*, 1886.]

I propose to consider in this paper the general problem of finding conics and cubics related to each other in such a manner that it may be possible to circumscribe about the conic an infinite number of polygons which are inscribed in the cubic. The idea of the greater part of this paper consists in deriving from a given conic three different cubics, which may be called the C, D, R cubics, respectively, and then showing that these cubics are related to the conic in the manner referred to. In the remaining part of the paper, I generate a nodal cubic from a given conic, and then show that the relation is satisfied for polygons of $2n$ sides, provided a condition depending on the number $2n$ is satisfied.

I present the first part in a form which has been suggested to me by Professor Cayley; and, in fact, I have incorporated herewith most of his report on this paper. Taking, for simplicity, the equation of the conic to be $xz-y^2=0$, the coordinates of a point thereof are as $1:\alpha:\alpha^2$, and the equation of the tangent at this point is

$$\alpha^2x-2\alpha y+z=0;$$

we may write then, in general, $t=\alpha^2x-2\alpha y+z$, that is,

$$t_1=\alpha_1^2x-2\alpha_1 y+z, \text{ \&c.},$$

so that $t_1=0$ is the tangent at the point α_1, and similarly t_2, t_3, &c.

First, considering any four tangents t_1, t_2, t_3, t_4, the C cubic is

$$C\equiv\lambda_1t_2t_3t_4+\lambda_2t_3t_4t_1+\lambda_3t_4t_1t_2+\lambda_4t_1t_2t_3=0,$$

that is,
$$\frac{\lambda_1}{t_1}+\frac{\lambda_2}{t_2}+\frac{\lambda_3}{t_3}+\frac{\lambda_4}{t_4}=0 \quad\text{......................(1)},$$

where λ_1, λ_2, λ_3, λ_4 are arbitrary coefficients.

Secondly, considering any six tangents t_1, t_2, t_3, t_4, t_5, t_6, the D cubic is
$$D\equiv t_1t_2t_3-\lambda t_4t_5t_6=0\text{..........................(2)},$$

where λ is an arbitrary coefficient.

Thirdly, consider any eight tangents t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, such

that the pairs t_1, t_5; t_2, t_6; t_3, t_7; t_4, t_8 meet in four points lying in a line. Let ϑ, k be arbitrary coefficients; and, writing for shortness $\vartheta_1 = \vartheta - a_1$, &c., we may establish between $a_1, a_2 \dots a_8$ the relations

$$\vartheta_1\vartheta_5 = \vartheta_2\vartheta_6 = \vartheta_3\vartheta_7 = \vartheta_4\vartheta_8 = k,$$

and, this being so, the equation of the line in question will be

$$(\vartheta^2 - k)\,x - 2\vartheta y + z = 0.$$

We have then, identically,

$$\vartheta_5\vartheta_6\vartheta_7\vartheta_8 t_1 t_2 t_3 t_4 - \vartheta_1\vartheta_2\vartheta_3\vartheta_4 t_5 t_6 t_7 t_8 = \left\{(\vartheta^2 - k)\,x - 2\vartheta y + z\right\} R \;\dots(3),$$

viz., the quartic function on the left hand will contain the linear factor $(\vartheta^2 - k)\,y - 2\vartheta y + z$, and the remaining factor will of course be a cubic function R.

The proof of this, given by Professor Cayley, is:—The quartic curve

$$\vartheta_5\vartheta_6\vartheta_7\vartheta_8 t_1 t_2 t_3 t_4 - \vartheta_1\vartheta_2\vartheta_3\vartheta_4 t_5 t_6 t_7 t_8 = 0$$

passes through the four points (t_1, t_5), (t_2, t_6), (t_3, t_7), (t_4, t_8), which lie on the line $(\vartheta^2 - k)\,x - 2\vartheta y + z = 0$; whence, if it passes through a fifth point of this line, the quartic will break up into this line and a cubic. But a point of the line in question is $x, y, z = 0, 1, 2\vartheta$, and for this point we have

$$t_1 = a_1^2 x - 2a_1 y + z = 2\,(\vartheta - a_1) = 2\vartheta_1, \text{ \&c.},$$

values which satisfy the equation of the quartic curve.

The general property is, that, given the conic and the cubic (C, D, or R, as the case may be), the system of tangents is not determinate, but possesses one degree of freedom, or, what is the same thing, may be considered as depending upon one arbitrary parameter. We can easily give a direct verification of this in each of the three cases. Let μ_1, μ_2 be the roots of the equation $a^2x - 2ay + z = 0$, then

$$t_1 = (a_1 - \mu_1)(a_1 - \mu_2), \text{ \&c.}$$

Putting these values of t_1, &c. in (1), and multiplying by $\mu_1 - \mu_2$, we get a result which may be written

$$\phi\,(\mu_1) - \phi\,(\mu_2) = 0.$$

This is satisfied by

$$\phi\,(\mu_1) - k = 0, \quad \phi\,(\mu_2) - k = 0,$$

that is, by two roots of the biquadratic

$$\phi\,(\mu) - k = 0;$$

from which it easily follows that the four tangents determined by the biquadratic form a quadrilateral having its six intersections of sides on the cubic. The conic is now easily seen to be, in reference to the cubic, one of those arrived at by Professor Cayley in an article in Liouville's *Journal de Mathématiques*, t. x., 1845, p. 102. This result evidently gives a solution for the case of triangles and quadrilaterals, and, in fact, as I have shown already (see § 12 of a paper entitled "On certain Results obtained by means of the Arguments of Points on a Plane Curve," *Proceedings*, Vol. xv., 1883, p. 4), constitutes the only solution of the problem of finding fixed conics touching the sides of an infinite number of triangles inscribed in a cubic.

Again, with respect to the second conic, we see that a similar transformation will bring it to the form

$$\phi(\mu_1)\,\phi(\mu_2)-\lambda=0,$$

which may also be written

$$\psi(\mu_1)\,\psi(\mu_2)-\lambda=0,$$

where
$$\psi(\mu)=\frac{\lambda+k\phi(\mu)}{k+\phi(\mu)}\quad\text{.......................... (4).}$$

We thus perceive the possibility of writing the same cubic in the form (4) in a singly infinite number of ways. This result was given by Darboux, in his work, " Sur une classe remarquable de Courbes et de Surfaces Algébriques" (Paris, 1873), and evidently gives a solution for the case of quadrilaterals and hexagons; for the nine points of intersection of the tangents t_1, t_2, t_3 with the tangents t_4, t_5, t_6 lie on the cubic, and may evidently be considered as forming nine quadrilaterals or six hexagons.

The third cubic, which I have arrived at myself, being transformed to μ_1, μ_2 coordinates, takes the same form as Darboux's, and, therefore, the result follows in the same way. This cubic solves the problem for the case of quadrilaterals, hexagons, and octagons; for, omitting the four points (t_1, t_5), &c., from the intersection of $t_1t_2t_3t_4=0$ with $t_5t_6t_7t_8=0$, the remaining twelve points can be arranged so as to form six quadrilaterals, sixteen hexagons, or three octagons inscribed in the cubic. Four of the hexagons, it may be observed, will, by Pascal's theorem, have their vertices on a conic, namely, those included in the intersection of $t_1t_2t_3=0$, with $t_5t_6t_7=0$, &c.

We may now proceed to consider the relations between the conic and the cubics more particularly, and then, by means of some of the results obtained, determine the conics corresponding to a given cubic C, D, or R. First, since the given conic is one of a system of conics

such that the C cubic is the locus of the vertex of a pencil in involution circumscribed about three of the system (see Professor Cayley's paper which I have referred to above), it is easy to see that the result of substituting differential symbols in the equation of the conic, and operating on a cubic of which C is the Hessian, vanishes; that is, if

$$U \equiv x^3+y^3+z^3+6mxyz = 0$$

is one of the cubics of which C is the Hessian, then the conic must be of the form

$$\alpha\,(m\lambda^2-\mu\nu)+\beta\,(m\mu^2-\nu\lambda)+\gamma\,(m\nu^2-\lambda\mu) = 0.$$

We see thus that if the cubic be given, the conic belongs to a system having a common tangential Jacobian, and that there are three distinct systems corresponding to the three cubics of which the given cubic is the Hessian.

We may now proceed to consider some properties of Darboux's conic and cubic. If $k+\phi\,(\mu)$ has a square factor, it is evident that there will be four corresponding values of μ which are determined by the Jacobian of the two binary cubics of which $\phi\,(\mu)$ is the ratio. In this case two of the lines such as t_5, t_6 coincide, and the curve may be written

$$t_1 t_2 t_3 - k t_5^2 t_4 = 0 \quad\text{.............................(5)},$$

showing that the lines t_1, t_2, t_3 are tangents whose points of contact lie on t_5. Hence we infer that the points of contact of the twelve tangents to the cubic which are touched by one of the conics lie by threes on four tangents of the conic.

Again, the equation (4) can be transformed into

$$\psi\,(\mu_1)+\psi\,(\mu_2) = 0 \quad\text{.........................(6)},$$

where

$$\psi\,(\mu) = \frac{\phi\,(\mu) \pm \sqrt{\lambda}}{\phi\,(\mu) \mp \sqrt{\lambda}},$$

and the reduction to this form is obviously unique. Writing, then,

$$\psi\,(\mu) = \frac{(\mu-a)(\mu-b)(\mu-c)}{(\mu-a')(\mu-b')(\mu-c')} = \frac{P}{Q} \quad\text{..............(7)},$$

it is evident, from (6), that the vertices of the two triangles abc, $a'b'c'$ lie on the cubic, as well as the points of contact. Hence we infer the theorem, that, if two triangles be circumscribed about a conic, their six vertices and six points of contact lie on a cubic. We can now find the locus of the vertices of the triangles such as $t_1 t_2 t_3$, $t_4 t_5 t_6$ in (2);

for the cubic locus of the vertices of the quadrilateral formed by the tangents μ_1, μ_2, μ_3, μ_4 will, from (6), be given by the equations

$$\psi(\mu_1)+\psi(\mu_2)=0, \quad \psi(\mu_2)+\psi(\mu_3)=0, \quad \psi(\mu_3)+\psi(\mu_4)=0,$$
$$\psi(\mu_4)+\psi(\mu_1)=0,$$

from which we see that the intersections of opposite sides will be given by $\psi(\mu_1)-\psi(\mu_3)=0, \quad \psi(\mu_2)-\psi(\mu_4)=0.$

Now $\psi(\mu_1)-\psi(\mu_3)$, being divided by $\mu_1-\mu_3$ and equated to zero, will give the equation of a conic, which is, consequently, the locus of the intersection of the opposite sides of the quadrilaterals formed out of the intersections of the lines t_1, t_2, t_3 with the lines t_4, t_5, t_6; but these points evidently are the vertices of the triangles $t_1t_2t_3$, $t_4t_5t_6$. If we call this latter conic Θ, and if S is the conic touching t_1t_2, &c., it is easy to see that Θ intersects S in the four points of contact of the lines passing through the points of contact of the twelve tangents of the cubic which touch S. Also, it may be observed, that the satellites of the four lines just alluded to are the common tangents of S and Θ.

Since all the triangles such as $t_1t_2t_3$, $t_4t_5t_6$ are circumscribed about the conic S and inscribed in the conic Θ, it readily follows that they must be also self-conjugate with regard to another fixed conic. Now, if we substitute differential symbols in the tangential equation of a conic and operate on the product of the equations of the sides of a self-conjugate triangle, the result will vanish, from which, by means of (1), we see that, if we operate similarly with the conic found above on the equation of the cubic, the result must vanish. But this, as I have remarked already, is a property of Professor Cayley's system of conics. We see thus that all the triangles such as $t_1t_2t_3$ are self-conjugate with regard to one of Cayley's conics of involution.

If the vertices of the triangle $t_1t_2t_3$ be joined to the points of contact of the opposite sides with S, we know that the joining lines pass through a point, which we may call the pole of the triangle; and, when the triangles vary, we can show that the locus of these poles is a conic. It can easily be proved that the parameters of the tangents which pass through the pole of the triangle are the roots of the Hessian of the cubic which gives the parameters of the sides. Now, from (7), the cubic giving the parameters of the sides of the triangle $t_1t_2t_3$ involves an indeterminate linearly, and therefore its Hessian involves the same quantity in the second degree. In fact, from (7), $t_1t_2t_3$ is determined by $P+\lambda Q=0$, and $t_4t_5t_6$ by $P-\lambda Q=0$.

It thus appears that the poles of the two triangles lie on the same conic, and, since the parameters (the λ's) of these points are equal

with opposite signs, we can easily see that the line joining them passes through a fixed point.

When the conics S and Θ are given, it is easily seen that the cubic still involves two arbitrary constants; and these two constants will evidently be determined by assigning the coordinates of a fixed point through which passes the line joining the poles of two triangles, such as $t_1t_2t_3$, $t_4t_5t_6$. We thus arrive at a general mode of generating a cubic. Let S and Θ be two conics connected by an invariant relation such that an infinite number of triangles can be circumscribed about S and inscribed in Θ; then, taking two triangles of the system so that the line joining their poles with regard to S passes through a fixed point, the locus of the intersection of the sides of one triangle with those of the other is a cubic.

We may now proceed to express some of the preceding results in an analytical form. Let the cubic referred to a canonical triangle be written as follows,

$$U \equiv x^3+y^3+z^3+6mxyz = 0,$$

then Cayley's conics of involution are

$$\Sigma \equiv \alpha\,(m\lambda^2-\mu\nu)+\beta\,(m\mu^2-\nu\lambda)+\gamma\,(m\nu^2-\lambda\mu) = 0 \;\ldots\ldots\ldots(8),$$

in tangential coordinates, as I have shown already. It may be observed that Σ is the polar conic of the line α, β, γ with regard to the curve

$$m\,(\lambda^3+\mu^3+\nu^3)-3\lambda\mu\nu = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

which is easily found to be a contravariant of U of the ninth degree in the coefficients, namely,

$$3TP-4SQ = 0,$$

where S, T are the invariants, and P, Q are Cayley's contravariants of the third and fifth degrees, respectively, in the coefficients.

Now, we have seen that S touches the sides of an infinite number of triangles which are self-conjugate with regard to a conic of the system Σ; and hence, by the mode of generation, that the points of contact on Σ of the common tangents S and Σ lie on their respective satellites. But, if $\lambda x+\mu y+\nu z = 0$ is a line, and $\lambda' x+\mu' y+\nu' z = 0$ is its satellite, it can be shown that

$$\left.\begin{aligned}\lambda' &= \lambda^4-2\lambda\,(\mu^3+\nu^3)-6m\mu^2\nu^2\\ \mu' &= \mu^4-2\mu\,(\nu^3+\lambda^3)-6m\nu^2\lambda^2\\ \nu' &= \nu^4-2\nu\,(\lambda^3+\mu^3)-6m\lambda^2\mu^2\end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots(10).$$

Expressing, then, that a line L touches Σ, and that L and its satellite are conjugate with regard to Σ, we get

$$\alpha\,(m\lambda^2-\mu\nu)+\beta\,(m\mu^2-\nu\lambda)+\gamma\,(m\nu^2-\lambda\mu) = 0 \;\;\ldots\ldots\ldots(11),$$

$$\alpha\,(2m\lambda\lambda'-\mu\nu'-\mu'\nu)+\beta\,(2m\mu\mu'-\nu\lambda'-\nu'\lambda)+\gamma\,(2m\nu\nu'-\lambda\mu'-\lambda'\mu) = 0 \;\;\ldots\ldots\ldots(12),$$

the latter of which equations becomes, after substituting for λ', μ', ν', from (10),

$$\alpha\,(A\lambda^2-B\mu\nu)+\beta\,(A\mu^2-B\nu\lambda)+\gamma\,(A\nu^2-B\lambda\mu) = 0 \;\;\ldots\ldots(13),$$

where
$$A \equiv 2m\,(\lambda^3+\mu^3+\nu^3)+\lambda\mu\nu,$$
$$B \equiv \lambda^3+\mu^3+\nu^3+4m^2\lambda\mu\nu.$$

Hence, from (11) and (13), since $A-mB$ cannot vanish in general, we must have

$$\alpha\lambda^2+\beta\mu^2+\gamma\nu^2 = 0, \quad \alpha\mu\nu+\beta\nu\lambda+\gamma\lambda\mu = 0,$$

from which we see that

$$\alpha\lambda^2+\beta\mu^2+\gamma\nu^2+k\,(\alpha\mu\nu+\beta\nu\lambda+\gamma\lambda\mu) = 0$$

represents a conic touching four tangents of Σ at their intersection with their satellites, and will therefore coincide with S if k be properly determined. This is done at once by expressing the invariant condition that S is inscribed in triangles self-conjugate with regard to Σ. We thus find that the tangential equation of S is

$$\left\{2m\,(\alpha^3+\beta^3+\gamma^3)+3\alpha\beta\gamma\right\}\,(\alpha\lambda^2+\beta\mu^2+\gamma\nu^2)$$
$$+(\alpha^3+\beta^3+\gamma^3-12m^2\alpha\beta\gamma)(\alpha\mu\nu+\beta\nu\lambda+\gamma\lambda\mu) = 0\ldots(14),$$

which may be transformed into

$$3P\,(48S^2P'+TQ')+Q\,(3TP'-4SQ') = 0 \;\;\ldots\ldots\ldots\ldots(15),$$

where P, Q are the results of substituting the coordinates of a fixed line in P, Q, and P', Q' are the polar conics of the same line with regard to P, Q, respectively. This, then, is the general equation of the system of conics inscribed in pairs of triangles such as $t_1t_2t_3$, $t_4t_5t_6$, whose nine intersections of sides lie on the cubic. We can now find the equation of the conic which we have called Θ. From the fact that Θ circumscribes all triangles which are simultaneously self-conjugate with regard to Σ and circumscribed about S, we readily find that Θ is the harmonic covariant of S and Σ, that is, the locus

of points whence the tangents to S and Σ form a harmonic pencil. We thus obtain

$$\begin{aligned}\Theta \equiv & \left\{\alpha^3+\beta^3+\gamma^3-12m^2\alpha\beta\gamma\right\} \\ & \times \left\{4m\beta\gamma x^2+4m\gamma\alpha y^2+4m\alpha\beta z^2+2\alpha^2 yz+2\beta^2 zx+2\gamma^2 xy\right\} \\ & +\left\{2m\,(\alpha^3+\beta^3+\gamma^3)+3\alpha\beta\gamma\right\} \\ & \times \left\{\alpha^2x^2+\beta^2y^2+\gamma^2z^2-2\,(m\alpha^2+\beta\gamma)\,yz-2\,(m\beta^2+\gamma\alpha)\,zx\right. \\ & \qquad\qquad \left.-2\,(m\gamma^2+\alpha\beta)\,xy\right\}=0 \quad\ldots(16).\end{aligned}$$

It may be observed that, when the conic (15) breaks up into factors, the two points so represented are corresponding points on the Hessian, and we can show, otherwise, that we can draw through two such points triads of lines intersecting in nine points on the curve. Writing a cubic in the form

$$U \equiv \alpha x^3+\beta y^3+\gamma z^3+\delta u^3 = 0,$$

where

$$x+y+z+u=0,$$

the points xy, zu are corresponding points on the Hessian. The curve is then evidently the locus of the intersection of the system of lines

$$\alpha x^3+\beta y^3+\lambda\,(x+y)^3=0,$$

passing through xy, with the system

$$\gamma z^3+\delta u^3+\lambda\,(z+u)^3=0,$$

passing through zu.

We may now proceed to consider the case in which the equation (1) represents a cubic with a double point. This will evidently occur if the curve is capable of being transformed so as to become of the form

$$t_1^2 t_3-\lambda t_4^2 t_5=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

for then the point t_1, t_4 is a node. Now, these tangents t_1, t_4 are determined by the Jacobian J of the cubics P, Q (7), from which we see that the node must be one of the six intersections of sides of the quadrilateral formed by the four tangents corresponding to the roots of J, which four lines, as we have seen already, are the common tangents of S and Σ. This result may be also stated in the following manner:—If two triangles are both circumscribed about a conic S, their six critic centres are at the intersections of the sides of the quadrilateral formed by the common tangents of S and the conic with regard to which the triangles are self-conjugate.

It may be observed that, when the cubic is given in the form (6)

and (7), the condition for a double point is found by expressing that the biquadratic in λ determined by the discriminant of $P+\lambda Q$ has two roots equal with opposite signs.

We see now, from (17), that if a cubic have a node O, and A, B are the points of contact of the tangents drawn from a point C on the curve, then any conic touching the lines OA, OB, CA, CB is of the kind which we have been discussing.

If in (17) t_3 coincides with t_1, the curve becomes

$$t_1^3 - \lambda t_4^2 t_5 = 0,$$

and is, therefore, cuspidal. Hence in this case all the conics are inscribed in the triangle formed by the cuspidal tangent, the inflexional tangent, and the connector of the points of contact of these lines. If the curve be written $y^3 - x^2 z = 0$, and the conic S be expressed in the form

$$\sqrt{lx} + \sqrt{my} + \sqrt{nz} = 0 \quad\text{.......................(18)},$$

the corresponding conic Θ will be

$$(lx - nz)^2 - mnyz = 0.$$

I now proceed to consider the third cubic R more particularly. Taking z as the line on which the tangents t_1, t_5, &c. intersect, and writing the conic $z^2 - 4xy = 0$, it is evident that each of these pairs of tangents will have equal parameters with opposite signs. It is easy to see, then, that the equation (3) can be transformed into

$$\psi(\mu_1) + \psi(\mu_2) = 0 \quad\text{..........................(19)},$$

where
$$\psi(\mu) = \frac{\mu^4 + a\mu^2 + b}{\mu(\mu^2 + c)},$$

in which case the tangents $t_1 t_2 t_3 t_4$ are determined by

$$\mu^4 + a\mu^2 + b + k\mu(\mu^2 + c) = 0 \quad\text{..................(20)},$$

and $t_5 t_6 t_7 t_8$ by the same equation with the sign of k changed, k being a variable parameter. It follows, hence, that when two of the lines in the first quartet coincide, so also will two in the second, the curve then being written

$$t_1^2 t_3 t_4 - k t_5^2 t_7 t_8 = 0 \quad\text{............................(21)},$$

and, since the discriminant of (20) is a cubic for k^2, there are three forms of (21), which give the twelve common tangents of the conic and cubic, $t_3 t_4$, $t_7 t_8$, &c. Now, the point xy, namely, the pole of z with regard to the conic, lies on the cubic, and lines passing through this point are evidently divided harmonically by z and the cubic, from

which it follows that xy is a point of inflexion, and z its corresponding harmonic polar. But, from (21), t_1, t_5 is a node of the complex curve, that is, one of the points where the cubic is intersected by the line z. We see thus, from (21), that six chords of contact of the tangents of the cubic which are touched by the conic intersect by pairs at the three points where the cubic is met by the harmonic polar of a point of inflexion.

Hence, from what we have proved, there are nine distinct systems of conics of the kind we have been investigating corresponding to the nine points of inflexion, and, being given a point of inflexion, we can find one of the conics as follows:—Let O be one of the points where the corresponding harmonic polar L meets the curve, then, drawing two lines through O harmonically connected with L and the tangent at O to meet the curve again in A, B, C, D, the conic touching these lines and the tangents to the cubic at A, B, C, D, is the conic required. We see thus that, when the cubic is given, the conic involves one arbitrary parameter.

To find the locus of intersections of the opposite sides of the quadrilaterals and of the alternate sides of the hexagons, &c., in this case, we observe that these points are the six intersections of sides of each of the two quadrilaterals $t_1t_2t_3t_4$, $t_5t_6t_7t_8$. Hence, from (19), we find for this locus

$$\psi(\mu_1)-\psi(\mu_2)=0,$$

which, being divided by $\mu_1-\mu_2$, represents a cubic with regard to which the line z is the harmonic polar of the point of inflexion xy. In fact, if we put

$$\mu_1+\mu_2 : \mu_1\mu_2 : 1 = z : y : x,$$

in (20), the equation of the given curve is

$$(bx+cy)(z^2-3xy)+y^3+bxy^2+acx^2y+bcx^3=0,$$

and that of the cubic which we have just determined is

$$(cy-bx)(z^2-xy)+y^3-bxy^2+acx^2y-bcx^3=0.$$

If the invariants S and T of the biquadratic (20) both vanish, that expression has a cube factor, and (21) becomes reducible to the form

$$t_1^3\,t_4-t_5^3\,t_6=0 \qquad\qquad (22),$$

which, being divided by z, represents a nodal cubic of which t_4, t_6 are inflexional tangents, and t_1, t_5 the lines joining their points of contact to the node. Hence, in the case of a cubic with a node O, if I, I' are two points of inflexion, then every conic of the system touches

the tangents at I, I' and the lines OI, OI'. There are thus, as we see, in this case three distinct systems of conics corresponding to the three pairs of points of inflexion.

It may be remarked that there are no conics of this system in the case of cubics with a cusp.

In the preceding cases I have considered conics which constitute solutions of the problem when the number n of the sides of the polygon takes the values 3, 4, 6, 8, successively; and, in the case of the general cubic, I have not been able to come upon any other conics which constitute solutions for these or greater values of n. For the nodal cubic, however, as I have remarked at the commencement of this paper, I have found a system of conics involving a single parameter, which give solutions for even values of n, certain determinate values of the parameter corresponding to each value of n.

I now proceed to investigate this system. It is known that the integral

$$\int \frac{d\mu}{\{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)\}^{\frac{2}{3}}}$$

is an elliptic integral of the first kind with a numerical modulus, and in fact it is a particular case of Abel's theorem that the differential equation

$$\frac{d\mu_1}{\{(\mu_1-\alpha)(\mu_1-\beta)(\mu_1-\gamma)\}^{\frac{2}{3}}}+\frac{d\mu_2}{\{(\mu_2-\alpha)(\mu_2-\beta)(\mu_2-\gamma)\}^{\frac{2}{3}}}=0 \quad \ldots(23)$$

admits of an algebraic integral which may be written in the form

$$(\beta-\gamma)\sqrt[3]{\{(\alpha-\mu_1)(\alpha-\mu_2)(\alpha-k)\}}+(\gamma-\alpha)\sqrt[3]{\{(\beta-\mu_1)(\beta-\mu_2)(\beta-k)\}}$$
$$+(\alpha-\beta)\sqrt[3]{\{(\gamma-\mu_1)(\gamma-\mu_2)(\gamma-k)\}}=0 \quad \ldots\ldots\ldots\ldots(24),$$

where k is a constant introduced by integration.

Considering now the conic which is the envelope of the line

$$\mu^2x-2\mu z+y=0,$$

if X, Y, Z are the tangents corresponding to the values α, β, γ of the parameter μ, we may write

$$X=(\beta-\gamma)^2(\alpha-\mu_1)(\alpha-\mu_2), \quad Y=(\gamma-\alpha)^2(\beta-\mu_1)(\beta-\mu_2),$$
$$Z=(\alpha-\beta)^2(\gamma-\mu_1)(\gamma-\mu_2).$$

The conic itself, it is easy to see, may then be written

$$X^{\frac{1}{2}}+Y^{\frac{1}{2}}+Z^{\frac{1}{2}}=0,$$

and the equation (23) becomes

$$(lX)^{\frac{1}{3}}+(mY)^{\frac{1}{3}}+(nZ)^{\frac{1}{3}}=0 \quad\text{.....................}(25),$$

where

$$l=(\alpha-k)(\beta-\gamma),\ m=(\beta-k)(\gamma-\alpha),\ n=(\gamma-k)(\alpha-\beta)\text{...}(26),$$

and, therefore, $\qquad l+m+n=0.$

Putting, now, x, y, z for lX, mY, nZ, respectively, this result may be stated as follows:—Two tangents to the conic

$$\left(\frac{x}{l}\right)^{\frac{1}{2}}+\left(\frac{y}{m}\right)^{\frac{1}{2}}+\left(\frac{z}{n}\right)^{\frac{1}{2}}=0\text{.....................}(27),$$

where $$l+m+n=0 \quad\text{..........................}(28),$$

whose parameters μ_1, μ_2 are connected by the equation (23), intersect on the cubic $$x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0 \quad\text{..........................}(29).$$

Since, from (29), the lines x, y, z are the inflexional tangents of the cubic, we see that the conic touches these three lines; also, taking the envelope of (27) subject to the condition (28), we get the cubic (29), showing that the conic touches the cubic at the point l^3, m^3, n^3, or the point of contact of the tangent whose parameter is the constant k in (24).

Putting $$u=\int\frac{d\mu}{\{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)\}^{\frac{2}{3}}},$$

we have, from (22), by integration,

$$u_1+u_2=\sigma,$$

where σ is a constant, which may be multiplied by a cube root of unity. Hence, for a polygon with an even number of sides n, we have

$$u_1+u_2=\epsilon_1\sigma,\quad u_2+u_3=\epsilon_2\sigma,\quad\text{......},\quad u_n+u_1=\epsilon_n\sigma\quad\text{......}(30),$$

where ϵ_1, ϵ_2, &c. are values of the cube root of unity. We see thus that there will be an infinite number of such polygons, provided we have $$\sigma\,(\epsilon_1-\epsilon_2+\ldots-\epsilon_n)=\omega \quad\text{.........................}(31),$$

where ω is a complete value of the integral u.

In the case of the quadrilateral, the only relevant value of σ is found to be determined by the equation $3\sigma=\omega$. Hence, from (23), we have

$$\epsilon_1 l+\epsilon_2 m+\epsilon_3 n=0\quad\text{or}\quad l^2+m^2+n^2=0,$$

from which we see that the values of k are given by the Hessian of

the cubic whose roots are α, β, γ. It is easy to see that both these conics are touched by the line $x+y+z=0$, namely, the line of contact of the inflexional tangents. If the curve then be written in the form

$$ABC-D^3=0,$$

the two conics touch the lines A, B, C, D, and, therefore, belong to Darboux's system, so that we have obtained no new conics in this case. In the case of the hexagon, the relevant forms of (31) come under the two equations

$$(4\epsilon_1-\epsilon_2)\,\sigma=\omega \qquad\qquad (32),$$

$$(3\epsilon_1-3\epsilon_2)\,\sigma=\omega \qquad\qquad (33).$$

The first of these equations (32) may be considered as resulting from the elimination of σ' between

$$2\epsilon_1\sigma+\sigma'=\omega, \quad \epsilon_2\sigma+2\sigma'=\omega.$$

If, then, k, k' are the values of μ corresponding to the values σ, σ' of u, we have, from (24),

$$\Sigma\,(\beta-\gamma)\ \sqrt[3]{\{(\alpha-k)^2(\alpha-k')\}}=0 \qquad\qquad (34),$$

$$\Sigma\,(\beta-\gamma)\ \sqrt[3]{\{(\alpha-k)(\alpha-k')^2\}}=0 \qquad\qquad (35).$$

But (34) being considered as a cubic for k', two of the roots coincide with k, and for the other we easily find

$$l':m':n'=l\,(m-n)^3:m\,(n-l)^3:n\,(l-m)^3,$$

where l, m, n, l', m', n' are the functions of k, k' respectively given at (26). Substituting, then, these values in (35), we get

$$\epsilon_1 l\,(m-n)^2+\epsilon_2 m\,(n-l)^2+\epsilon_3 n\,(l-m)^2=0 \qquad\qquad (36).$$

If we take $\epsilon_1=\epsilon_2=\epsilon_3$, the result obtained from this equation is irrelevant, namely, $lmn=0$. Taking $\epsilon_1=\epsilon_2$, we obtain twelve relevant solutions, after omitting the irrelevant factors l, m, n; and taking ϵ_1, ϵ_2, ϵ_3, all different, we get six solutions, amongst which are included the two conics just found for the case of the quadrilateral. We thus obtain, from (32), sixteen conics which constitute solutions in the case of the hexagon, and do not belong to any of the systems previously obtained.

For the octagon the relevant solutions are found from the equations

$$(4\epsilon_1-4\epsilon_2)\,\sigma=\omega \qquad\qquad (37),$$

$$(5\epsilon_1-2\epsilon_2)\,\sigma=\omega \qquad\qquad (38),$$

$$6\sigma=\omega \qquad\qquad (39).$$

From the last of these equations (39), we should find

$$(l-m)(m-n)(n-l) = 0,$$

that is, the parameters corresponding to k are determined by the covariant G of the cubic whose roots are α, β, γ. Now, each of these conics, so determined, touches the lines drawn from the node to two points of inflexion, and, therefore, belongs to one of the systems already investigated, see (22). The other two equations, (37) and (38), will evidently determine several conics which do not belong to the system just referred to; and, of course, for polygons with a greater number of sides, none of the conics obtained belong to any of the previously determined systems.

Thursday, March 11th, 1886.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Mr. L. J. Rogers was admitted into the Society.

The following communications were made :—

On an Instantaneous Proof of the Expression for the Number of Linearly Independent Invariants or Seminvariants of a given Type, and also of the Corresponding Expression for Reciprocants : Prof. J. J. Sylvester, F.R.S.

On Ternary and n-ary Reciprocants : E. B. Elliott, M.A.

Homographic, Circular, and Projective Reciprocants : L. J. Rogers, B.A.

A Proof of Cayley's Fundamental Theorem of Invariants : Captain MacMahon, R.A.

Note on the Invariantisers of a Binary Quantic : J. Griffiths, M.A.

The following presents were received :—

"Proceedings of the Royal Society," Vol. XXXIX., No. 240.

"Educational Times," for March.

"Bulletin des Sciences Mathématiques,—Tables des Matières et noms d'auteurs," T. IX., 1885.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. II., Fasc. 2 and 3.

"Beiblätter zu den Annalen der Physik und Chemie," B. x., St. 2.

"Sitzungsberichte der k. Preussischen Akad. der Wissenschaften zu Berlin," XL. to LII.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VI., No. 5.

"Sitzungsberichte der Physikalisch-medicinischen Societät zu Erlangen," 17 Heft, 1 Oct. 1884 bis 1 Oct. 1885.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," Mathematisch-physische Classe, 1885, III.; Leipzig, 1886.

"American Journal of Mathematics," Vol VIII., No. 2.

On Ternary and n-ary Reciprocants. *By* E. B. Elliott.

[*Read March* 11*th*, 1886.]

A. *Ternary Reciprocants.*

1. It is supposed that there are three variables, z, x, y, connected by a single relation.

An *absolute ternary reciprocant* is a function of the partial differential coefficients $\frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dx^2}, \frac{d^2z}{dx\,dy}, \frac{d^2z}{dy^2}, \dots$ of z with regard to x and y, which is equal to the altered function when z, x, y are interchanged cyclically, *i.e.*, to the same function of $\frac{dx}{dy}, \frac{dx}{dz}, \frac{d^2x}{dy^2}, \frac{d^2x}{dy\,dz}, \frac{d^2x}{dz^2}, \dots$, but for a factor which is constant and a cube root of unity. A wider definition, sometimes convenient, allows also the explicit introduction of the variables z, x, y themselves, these having to be interchanged cyclically in producing the altered function as well as the dependent and independent variables in the differential coefficients. Whenever, in what follows, such explicit introduction is contemplated, the fact will be specially stated.

Denote by A any such function, by A' the function with x as dependent variable obtained by a cyclical interchange of z, x, y, and by A'' the function with y as independent variable derived by a second cyclical interchange. Our definition supposes that either

$$A = A' = A'',$$

or

$$A = \omega A' = \omega^2 A'',$$

or

$$A = \omega^2 A' = \omega A'',$$

where $\omega = \frac{1}{2}(-1+\sqrt{-3})$. There are then three distinct classes of absolute ternary reciprocants, distinguished by the power of ω, which multiplying A' produces A. Let us speak of these three classes as of characters 0, 1, 2 respectively.

2. Let us use p, q to denote $\frac{dz}{dx}$ and $\frac{dz}{dy}$, p', q' to denote $\frac{dx}{dy}$ and $\frac{dx}{dz}$, the partial differential coefficients obtained by one cyclical interchange, and p'', q'' to denote $\frac{dy}{dz}, \frac{dy}{dx}$.

It is convenient to give the name of *ternary reciprocants* not only to unctions defined as absolute ternary reciprocants above, but, as in

Professor Sylvester's theory of binary reciprocants, to functions which upon multiplication by a simple function of the derivatives become absolute reciprocants. This factor function is, in fact, always a positive or negative power of pq. Thus, a function R of the derivatives with z as dependent variable is defined as a ternary reciprocant if a constant θ can be found, such that

$$\frac{R}{(pq)^{\frac{1}{3}\theta}} = \omega^{\kappa} \frac{R'}{(p'q')^{\frac{1}{3}\theta}} = \omega^{2\kappa} \frac{R''}{(p''q'')^{\frac{1}{3}\theta}},$$

the number of accents indicating which variable is taken as the dependent one, as before. The propriety of the fractional form given to the index of pq will be apparent later.

3. The convention of cyclical interchange which has here been, and will continue to be, adopted for clearness of statement, is not altogether necessary. It will be clear, in fact, that any reciprocant (the word ternary is often omitted where no confusion can arise), whether absolute or not, will have the property of reciprocance also for non-cyclical interchanges, if only, in addition to the fundamental property as above, it has the further one of involving x- and y-derivatives of z quite symmetrically. Such reciprocants have a special claim to the name, and may be designated *symmetrical* or *reversible* reciprocants, other reciprocants being irreversible. A reversible ternary reciprocant of character zero, made absolute by a power of pq, will then have six equivalent forms, which, however, are the same in pairs, viz.,

$$A^z_{xy} = A^z_{yx} = A^x_{yz} = A^x_{zy} = A^y_{zx} = A^y_{xz}.$$

4. As first examples, it may be well to mention three linear functions of the variables themselves without derivatives which possess the property of reciprocance. Whether themselves entitled to the designation of reciprocants or not, they are at any rate very valuable, just as true reciprocants are, as sources of other reciprocants. The three are

$$z+x+y = x+y+z = y+z+x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$z+\omega x+\omega^2 y = \omega\,(x+\omega y+\omega^2 z) = \omega^2\,(y+\omega z+\omega^2 x) \ldots\ldots\ldots (2),$$

$$z+\omega^2 x+\omega y = \omega^2\,(x+\omega^2 y+\omega z) = \omega\,(y+\omega^2 z+\omega x) \ldots\ldots\ldots (3),$$

whose characters are 0, 1, 2, respectively. Regarded as reciprocants they are absolute.

The last two of these three are not reversible. They enable us, however, to illustrate the way in which irreversible reciprocants may

be made to produce, by their combination, other reciprocants that are reversible. We notice the facts, that the sum of two absolute reciprocants of like character is an absolute reciprocant of that same character, and that the product of any number of reciprocants is a reciprocant whose character is the residue (mod. 3) of the sum of their characters; and, taking the product of the second and third of the above reciprocants, and the sum of the cubes of all three, we obtain

$$x^2+y^2+z^2-yz-zx-xy,$$

and

$$x^3+y^3+z^3+6xyz,$$

as absolute, and from their symmetrical forms reversible, reciprocants of character zero. These, by means of (1), may be replaced by the simpler pair

$$yz+zx+xy,$$

and

$$xyz.$$

It is, in fact, clear *à priori*, that all symmetric functions of x, y, and z have just the same right as $x+y+z$ to be regarded as reciprocantive. The linearity of the system (1), (2), (3) makes of those, even though two of them are irreversible, the most useful condensation of the aggregate of these quasi-reciprocants.

5. The equations connecting p, q with p', q', and with p'', q'', may be found by identification of the relations

$$dz = p\,dx+q\,dy,$$
$$dx = p'\,dy+q'\,dz,$$
$$dy = p''dz+q''dx,$$

which are equivalent expressions of the one connection between simultaneous infinitesimal increments of x, y, and z. From the first two we obtain

$$\frac{1}{-q'} = \frac{p}{-1} = \frac{q}{p'} = \left(\frac{pq}{p'q'}\right)^{\frac{1}{3}} \dots\dots\dots\dots\dots\dots\dots (4),$$

and from the second and third, and third and first, we get similar equalities, which can be written down at once by cyclical interchange of unaccented, singly accented, and doubly accented letters.

Now, from equations (4), we at once derive that

$$\left(\frac{pq}{p'q'}\right)^{\frac{1}{3}} = \frac{p+q-1}{p'+q'-1} = \frac{p+\omega q-\omega^2}{\omega\,(p'+\omega q'-\omega^2)} = \frac{p+\omega^2 q-\omega}{\omega^2\,(p'+\omega^2 q'-\omega)}$$
$$= \left\{\frac{S_r\,(p,\,q,\,-1)}{S_r\,(p',\,q',\,-1)}\right\}^{1/r} \dots\dots\dots\dots\dots\dots\dots (5),$$

where $S_r(\ldots)$ denotes any homogeneous symmetric function of degree r of its three arguments.

We see, then, that there are three reciprocants linear in $p, q, -1$ which involve p and q only, and that they are of characters 0, 1, 2 respectively, the equalities expressive of their reciprocance being

$$\frac{p+q-1}{(pq)^{\frac{1}{2}}} = \frac{p'+q'-1}{(p'q')^{\frac{1}{2}}} = \frac{p''+q''-1}{(p''q'')^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$\frac{p+\omega q-\omega^2}{(pq)^{\frac{1}{2}}} = \omega\frac{p'+\omega q'-\omega^2}{(p'q')^{\frac{1}{2}}} = \omega^2\frac{p''+\omega q''-\omega^2}{(p''q'')^{\frac{1}{2}}} \quad\ldots\ldots\ldots(7),$$

$$\frac{p+\omega^2 q-\omega}{(pq)^{\frac{1}{2}}} = \omega^2\frac{p'+\omega^2 q'-\omega}{(p'q')^{\frac{1}{2}}} = \omega\frac{p''+\omega^2 q''-\omega}{(p''q'')^{\frac{1}{2}}} \quad\ldots\ldots\ldots(8).$$

This system of three linear reciprocants embodies the whole of the class suggested by the last member of (5), *i.e.*, the class where reciprocance is expressed by

$$(pq)^{-\frac{1}{2}r}S_r(p,q,-1) = (p'q')^{-\frac{1}{2}r}S_r(p',q',-1) = (p''q'')^{-\frac{1}{2}r}S_r(p'',q'',-1) \quad\ldots\ldots\ldots\ldots\ldots(9);$$

but one of these is of sufficient geometrical importance, in connection with those ternary reciprocants which, following an analogy with certain binary reciprocants, may be called orthogonal, to receive special mention, viz.,

$$(pq)^{-1}(p^2+q^2+1) = (p'q')^{-1}(p'^2+q'^2+1) = (p''q'')^{-1}(p''^2+q''^2+1) \quad\ldots\ldots\ldots\ldots\ldots(10).$$

From equations (4), other immediate consequences are that

$$\left(\frac{pq}{p'q'}\right)^{\frac{1}{2}} = \frac{px+qy-z}{p'y+q'z-x} = \frac{\omega px+\omega^2 qy-z}{\omega(\omega p'y+\omega^2 q'z-x)} = \frac{\omega^2 px+\omega qy-z}{\omega^2(\omega^2 p'y+\omega q'z-x)}$$

$$= \left\{\frac{S_r(px,qy,-z)}{S_r(p'y,q'z,-x)}\right\}^{1/r} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Hence we have also three reciprocants (if we take the definition allowing the variables to enter explicitly) linear in px, qy, and z,

$$(pq)^{-\frac{1}{2}}(px+qy-z) = (p'q')^{-\frac{1}{2}}(p'y+q'z-x)$$
$$= (p''q'')^{-\frac{1}{2}}(p''z+q''x-y) \quad\ldots\ldots\ldots(12),$$

$$(pq)^{-\frac{1}{2}}(\omega px+\omega^2 qy-z) = \omega(p'q')^{-\frac{1}{2}}(\omega p'y+\omega^2 q'z-x)$$
$$= \omega^2(p''q'')^{-\frac{1}{2}}(\omega p''z+\omega^2 q''x-y) \ldots(13),$$

$$(pq)^{-\frac{1}{2}}(\omega^2 px+\omega qy-z) = \omega^2(p'q')^{-\frac{1}{2}}(\omega^2 p'y+\omega q'z-x)$$
$$= \omega(p''q'')^{-\frac{1}{2}}(\omega^2 p''z+\omega q''x-y) \ldots(14);$$

which between them are the equivalent of an entire system

$$(pq)^{-\frac{1}{3}r}\, S_r\,(px,\ qy,\ -z) = (p'q')^{-\frac{1}{3}r}\, S_r\,(p'y,\ q'z,\ -x)$$
$$= (p''q'')^{-\frac{1}{3}r}\, S_r\,(p''z,\ q''x,\ -y) \ \ldots(15).$$

In (13) to (16) we may write, instead of x, y, z, where they occur explicitly, $f(x, y, z)$, $f(y, z, x)$, $f(z, x, y)$ respectively, where f may be any function whatever.

A simple remark bearing upon the theory of envelopes may be made here. The singular solution, or aggregate of singular solutions, of the differential equation obtained by equating to zero any function of x, y, z, p, q which satisfies the law of reciprocance and is reversible, must involve x, y, z symmetrically. In particular, for example, the

envelope of $$z = cx + c'y + S_1(c,\ c',\ -1),$$

where $S_1(\ldots)$ is symmetrical, homogeneous, and of the first degree, but otherwise general, is a surface whose equation is symmetrical in x, y, and z.

6. A pair of absolute ternary reciprocants, involving p and q only in logarithmic form, may be found as follows. From

$$p = \frac{1}{q'},\quad q = -\frac{p'}{q'} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

we deduce $$pq^{\omega} = (-1)^{\omega} p'^{\omega} q'^{-1-\omega} = (-1)^{\omega} p'^{\omega} q'^{\omega^2},$$

whence $$pq^{\omega}\,(-1)^{\omega^2} = -p'^{\omega} q'^{\omega^2};$$

or, taking logarithms,

$$\log p + \omega \log q + \omega^2 \log(-1) = \omega\left\{\log p' + \omega \log q' + \omega^2 \log(-1)\right\}$$
$$= \omega^2\left\{\log p'' + \omega \log q'' + \omega^2 \log(-1)\right\} \ldots\ldots\ldots\ldots(17).$$

Also, in exactly the same manner,

$$\log p + \omega^2 \log q + \omega \log(-1) = \omega^2\left\{\log p' + \omega^2 \log q' + \omega \log(-1)\right\}$$
$$= \omega\left\{\log p'' + \omega^2 \log q'' + \omega \log(-1)\right\} \ldots\ldots\ldots\ldots(18).$$

The usefulness of $\log t$, in Professor Sylvester's theory of binary reciprocants, would lead us to expect these two absolute ternary reciprocants to be of great importance. The indeterminate imaginary constant $\log(-1)$ occurring in each will give little trouble, as it will disappear upon any differentiation. The expressions of reciprocance

(17) and (18) may however, if preferred, be written in real shape as follows:—

$$\log(p^2)+\omega\log(q^2) = \omega\left\{\log(p'^2)+\omega\log(q'^2)\right\}$$
$$= \omega^2\left\{\log(p''^2)+\omega\log(q''^2)\right\},$$
$$\log(p^2)+\omega^2\log(q^2) = \omega^2\left\{\log(p'^2)+\omega^2\log(q'^2)\right\}$$
$$= \omega\left\{\log(p''^2)+\omega^2\log(q''^2)\right\}.$$

The characters of these two reciprocants are 1 and 2 respectively. From them can at once be produced two absolute reciprocants of character zero, by multiplication and by cubing and addition respectively, viz.,

$$\left\{\log(p^2)\right\}^2+\left\{\log(q^2)\right\}^2-\log(p^2)\log(q^2) \quad \text{.........(19)}$$

and
$$2\left\{\log(p^2)\right\}^3+2\left\{\log(q^2)\right\}^3-3\log(p^2)\log(q^2)\left\{\log(p^2)+\log(q^2)\right\} \quad \text{..............(20)},$$

of which the last may also be written

$$\left\{\log(p^2)+\log(q^2)\right\}\left\{2\log(p^2)-\log(q^2)\right\}\left\{\log(p^2)-2\log(q^2)\right\}.$$

The complexity of these makes them, however, less serviceable than their equivalents (17) and (18).

7. Before passing from the subject of reciprocants involving p and q only, we may see that, if u and v be two such absolute reciprocants, the Jacobian of u and v with regard to p and q is a reciprocant made absolute upon multiplication by the positive first power pq.

Using the values (16) for p and q in terms of p' and q', we see that

$$\frac{d(p,q)}{d(p',q')} = \begin{vmatrix} 0, & -\dfrac{1}{q'^2} \\ -\dfrac{1}{q'}, & \dfrac{p'}{q'^2} \end{vmatrix} = -\frac{1}{q'^3} = \frac{pq}{p'q'} \quad \text{............(21)}.$$

Hence, if α, β be the characters of the two absolute reciprocants u, v,

$$\frac{d(u,v)}{d(p,q)} = \omega^{\alpha+\beta}\frac{d(u',v')}{d(p',q')}\cdot\frac{d(p',q')}{d(p,q)} = \omega^{\alpha+\beta}\frac{p'q'}{pq}\cdot\frac{d(u',v')}{d(p',q')};$$

i.e., as was to be shown,

$$pq\frac{d(u,v)}{d(p,q)} = \omega^{\alpha+\beta}p'q'\frac{d(u',v')}{d(p',q')} = \omega^{2(\alpha+\beta)}p''q''\frac{d(u'',v'')}{d(p'',q'')} \quad \text{......(22)}.$$

The character of the deduced reciprocant is the residue (mod. 3) of the sum of the characters of u and v.

Analogy with binary reciprocants would lead us to expect that this is only a simple case of a much more general theorem.

8. Another Jacobian whose value will be most useful is that of x, y considered as functions of y, z. Now, remembering that

$$dx = p'dy + q'dz,$$

we see that $$\frac{d(x, y)}{d(y, z)} = \begin{vmatrix} p', & q' \\ 1, & 0 \end{vmatrix} = -q' = \left(\frac{p'q'}{pq}\right)^{\frac{1}{2}}.$$

As a first application, we notice that $\iint (pq)^{\frac{1}{2}}\, dx\, dy$ has the property of an absolute reciprocant; that, in fact, between corresponding limits

$$\int^x \int^y (pq)^{\frac{1}{2}}\, dx\, dy = \int^y \int^z (p'q')^{\frac{1}{2}}\, dy\, dz = \int^z \int^x (p''q'')^{\frac{1}{2}}\, dz\, dx \quad \ldots(23).$$

It seems not unlikely that this double integral may be a valuable reciprocant to use in generating others, as it is the direct analogue of the even binary reciprocant

$$\int^x \sqrt{t}\, dx = \int^y \sqrt{\tau}\, dy,$$

which, with the odd one $\log t = -\log \tau$, by means of the theorem that, ϕ and ψ being two absolute binary reciprocants, $\frac{d\phi}{dx} \div \frac{d\psi}{dx}$ is another, produces Professor Sylvester's series of fundamental educts. For the analogous purpose, however, as will be seen below, we need to consider the double integral above as the sum of the products of the elements of two absolute reciprocants whose Jacobian is $(pq)^{\frac{1}{2}}$ rather than as an irresoluble reciprocant; and the resolution in question I have been unable to effect.

A consequence of (23) may be added. If R be any absolute reciprocant of character κ, then

$$\iint (pq)^{\frac{1}{2}} R\, dx\, dy = \omega^{\kappa} \iint (p'q')^{\frac{1}{2}} R'\, dy\, dz = \omega^{2\kappa} \iint (p''q'')^{\frac{1}{2}} R''\, dz\, dx \ldots(24)$$

is another. Of this an important particular case is that of the "orthogonal" reciprocant

$$\iint (1+p^2+q^2)^{\frac{1}{2}}\, dx\, dy = \iint (1+p'^2+q'^2)^{\frac{1}{2}}\, dy\, dz = \iint (1+p''^2+q''^2)^{\frac{1}{2}}\, dz\, dx \quad \ldots\ldots\ldots\ldots\ldots(25),$$

as to which more will be said later.

9. Before considering the existence of reciprocants involving second and higher derivatives, it is necessary to introduce some additional notation.

Let $$a_1,\ b_1,\ c_1 \text{ denote } \frac{d^2z}{dx^2},\ \frac{d^2z}{dx\,dy},\ \frac{d^2z}{dy^2},$$

$$a_2,\ b_2,\ c_2,\ d_2 \text{ denote } \frac{d^3z}{dx^3},\ \frac{d^3z}{dx^2\,dy},\ \frac{d^3z}{dx\,dy^2},\ \frac{d^3z}{dy^3},\ \text{\&c. \&c.},$$

while $a_1',\ b_1',\ c_1',\ a_2',\ b_2',\ c_2',\ d_2',\ \ldots$ represent the cyclically derived differential coefficients of x with regard to y and z, and double accents refer in like manner to partial differentiation of y with regard to z and x. Each suffix is the weight of the element to which it is attached, *i.e.*, is its dimensions in magnitudes of the kind x^{-1}, y^{-1}, z^{-1}.

Now, when a function ϕ expressed with x and y as independent variables becomes ψ' upon expression with y and z as independent,

$$\frac{d\phi}{dx} = \frac{d\psi'}{dz}\cdot\frac{dz}{dx} = p\,\frac{d\psi'}{dz},$$

and $$\frac{d\phi}{dy} = \frac{d\psi'}{dy} + \frac{d\psi'}{dz}\cdot\frac{dz}{dy} = \frac{d\psi'}{dy} + q\,\frac{d\psi'}{dz}.$$

Hence, applying these facts repeatedly to the equalities of transformation,

$$\left.\begin{aligned} p &= \frac{1}{q'} \\ q &= -\frac{p'}{q'} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

we obtain

$$\left.\begin{aligned} a_1 &= -\frac{p}{q'^2}\,c_1' = -p^3c_1' \\ b_1 &= -p^2\,(b_1'+qc_1') \\ c_1 &= -p\,(a_1'+2qb_1'+q^2c_1') \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots(26);$$

and hence, remembering that

$$\frac{d}{dy}\,F(p',\,q') \quad \text{and} \quad \frac{d}{dz}\,F(p',\,q')$$

involve a_1', b_1', c_1' linearly, and have no terms free from those second derivatives,

$$\left.\begin{aligned} a_2 &= -p^4d_2' + \\ b_2 &= -p^3\,(c_2'+qd_2') + \\ c_2 &= -p^2\,(b_2'+2qc_2'+q^2d_2') + \\ d_2 &= -p\,(a_2'+3qb_2'+3q^2c_2'+q^3d_2') + \end{aligned}\right\} \qquad \ldots\ldots\ldots(27),$$

where the additional terms in each are of the second order in a_1', b_1', c_1',

$$\left.\begin{aligned} a_3 &= -p^5e_3'+ \\ b_3 &= -p^4(d_3'+qe_3')+ \\ c_3 &= -p^3(c_3'+2qd_3'+q^2e_3')+ \\ d_3 &= -p^2(b_3'+3qc_3'+3q^2d_3'+q^3e_3')+ \\ e_3 &= -p\,(a_3'+4qb_3'+6q^2c_3'+4q^3d_3'+q^4e_3')+ \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(28),$$

the additional terms in each involving products of two or more suffixed elements: and so on continually.

Inspection of (16) and (26) enables us readily to discover two reciprocant expressions, and to write, remembering that

$$-p = \left(\frac{pq}{p'q'}\right)^{\frac{1}{2}},$$

$$\frac{(1+q^2)\,a_1-2pqb_1+(1+p^2)\,c_1}{pq} = \frac{(1+q'^2)\,a_1'-2p'q'b_1'+(1+p'^2)\,c_1'}{p'q'}$$

$$= \frac{(1+q''^2)\,a_1''-2p''q''b_1''+(1+p''^2)\,c_1''}{p''q''}\quad\ldots\ldots\ldots\ldots(29),$$

and $$\frac{a_1c_1-b_1^2}{(pq)^{\frac{4}{3}}} = \frac{a_1'c_1'-b_1'^2}{(p'q')^{\frac{4}{3}}} = \frac{a_1''c_1''-b_1''^2}{(p''q'')^{\frac{4}{3}}}\quad\ldots\ldots\ldots\ldots(30).$$

The forms of the higher derivatives are, however, too complicated to allow mere inspection to conduct us any further.

The two absolute reciprocants here obtained are, as is known, of great geometrical importance. Multiplied, in fact, by the third and fourth powers respectively of the absolute reciprocant

$$(pq)^{\frac{1}{3}} \div (p^2+q^2+1)^{\frac{1}{2}},$$

they become the expressions for the sum and product of the principal curvatures at any point of a surface.

10. The ternary reciprocant $a_1c_1-b_1^2$ just found is *pure*, *i.e.*, it contains explicitly neither the variables nor the first derivatives p, q. One proposition as to pure reciprocants in general can here be given.

In the equalities (26), (27), (28), it will be seen that each unaccented suffixed element is given as a sum of linear and higher functions of the accented suffixed elements. Every homogeneous function of degree m of the unaccented ones is then equal to a function of the accented of which the lowest terms are of the m^{th} degree, and are exactly obtained by forming the same homogeneous function

of the linear parts of the values in terms of them of the unaccented. If the homogeneous function be a reciprocant, two things are therefore necessary. The terms of the $(m+1)^{\text{th}}$ and higher degrees must vanish identically; and the function of the linear members must be the same function of the accented suffixed coefficients themselves, but for a factor involving only p and q, and for a mark of character. The first of these necessities would doubtless upon investigation lead us to annihilators of pure ternary reciprocants. It is the second and simpler one which we proceed to interpret.

The values in (26) for $-\dfrac{1}{p}$ times a_1, b_1, and c_1 are respectively the coefficients of a^2, $2a\beta$, β^2 in

$$a_1'\beta^2+2b_1'\beta\,(pa+q\beta)+c_1'\,(pa+q\beta)^2;$$

the linear terms in the values for $-\dfrac{1}{p}$ times a_2, b_2, c_2, d_2 are the coefficients of a^3, $3a^2\beta$, $3a\beta^2$, β^3, respectively, in

$$a_2'\beta^3+3b_2'\beta^2\,(pa+q\beta)+3c_2'\beta\,(pa+q\beta)^2+d_2'\,(pa+q\beta)^3;$$

those in the values of $-\dfrac{1}{p}$ times a_3, b_3, c_3, d_3, e_3 are the coefficients of a^4, $4a^3\beta$, $6a^2\beta^2$, $4a\beta^3$, β^4 in

$$a_3'\beta^4+4b_3'\beta^3\,(pa+q\beta)+6c_3'\beta^2(pa+q\beta)^2+4d_3'\beta\,(pa+q\beta)^3+e_3'\,(pa+q\beta)^4;$$

and so on continually.

Our supposed homogeneous pure reciprocant would, then, equally be one if, as is not the case after the first stage, the identities of transformation were

$$-\frac{1}{p}\,(a_1,\,b_1,\,c_1)(a,\,\beta)^2=(a_1',\,b_1',\,c_1')(a',\,\beta')^2,$$

$$-\frac{1}{p}\,(a_2,\,b_2,\,c_2,\,d_2)(a,\,\beta)^3=(a_2',\,b_2',\,c_2',\,d_2')(a',\,\beta')^3,$$

$$-\frac{1}{p}\,(a_3,\,b_3,\,c_3,\,d_3,\,e_3)(a,\,\beta)^4=(a_3',\,b_3',\,c_3',\,d_3',\,e_3')(a',\,\beta')^4,$$

$$\&c.,\qquad\qquad \&c.,$$

where
$$a'=\beta,\ \text{and}\ \beta'=pa+q\beta,$$

of which linear transformation the modulus is $-p$, *i.e.* $\left(\dfrac{pq}{p'q'}\right)^{\frac{1}{3}}$. The same substitution should be made for $-p$ in the multiplier of the left-hand quantics above.

The conclusion which we have to draw is, that any homogeneous

pure ternary reciprocant of character zero is an invariant of the system of emanants

$$\left(a\frac{d}{dx}+\beta\frac{d}{dy}\right)^2 z, \quad \left(a\frac{d}{dx}+\beta\frac{d}{dy}\right)^3 z, \;\ldots\ldots,$$

regarded as binary quantics in a and β.

The converse, that every such invariant be a reciprocant, is very far indeed from being established, or indeed true.

The presence of $-\frac{1}{p}$, *i.e.* $\left(\frac{p'q'}{pq}\right)^{\frac{1}{2}}$ as a factor of the left-hand quantics above, makes the index of the power of the modulus $\left(\frac{p'q'}{pq}\right)^{\frac{1}{2}}$ in the expression of reciprocance different from the index of the allied invariant. If, in fact, the index of the power of the modulus which multiplying an invariant I of order m produces the same invariant of the transformed quantics be θ, and if R', the same function of the accented derivatives, be a reciprocant, the expression of reciprocance is at once seen to be

$$\frac{R}{(pq)^{\frac{1}{2}(\theta+m)}} = \frac{R'}{(p'q')^{\frac{1}{2}(\theta+m)}} = \frac{R''}{(p''q'')^{\frac{1}{2}(\theta+m)}} \ldots\ldots\ldots\ldots\ldots (31).$$

As an example, notice the Hessian (30) for which $\theta = 2$.

The results of the present article may also be exhibited as a consequence of the fact that, a, β, γ being any corresponding increments given to x, y, and z,

$$\gamma - pa - q\beta = \left(\frac{pq}{p'q'}\right)^{\frac{1}{2}}(a - p'\beta - q'\gamma) = \left(\frac{pq}{p''q''}\right)^{\frac{1}{2}}(\beta - p''\gamma - q''a) \ldots (32),$$

so that, expanding each member by Taylor's theorem,

$$\frac{1}{(pq)^{\frac{1}{2}}}\left\{\frac{1}{1.2}(a_1, b_1, c_1)(a, \beta)^2 + \frac{1}{1.2.3}(a_2, b_2, c_2, d_2)(a, \beta)^3 + \ldots\right\}$$

$$= \frac{1}{(p'q')^{\frac{1}{2}}}\left\{\frac{1}{1.2}(a_1', b_1', c_1')(\beta, \gamma)^2 + \frac{1}{1.2.3}(a_2', b_2', c_2', d_2')(\beta, \gamma)^3 + \ldots\right\}$$

$$= \frac{1}{(p''q'')^{\frac{1}{2}}}\left\{\frac{1}{1.2}(a_1'', b_1'', c_1'')(\gamma, a)^2 + \frac{1}{1.2.3}(a_2'', b_2'', c_2'', d_2'')(\gamma, a)^3 + \ldots\right\}$$

$$\ldots\ldots\ldots\ldots\ldots (33).$$

11. The generation of ternary reciprocants from others by eduction is a much less simple matter than the analogous generation of ordinary binary reciprocants. From an ordinary reciprocant an infinite series can, we know, be educed if any one other reciprocant independent of the first is known, and the processes of eduction are merely those

of simple successive differentiation. The allied processes as to ternary reciprocants require, however, previous knowledge, tacit or expressed, of at least three reciprocants, and the processes of derivation are naturally more complicated.

Perhaps the simplest of the theorems of ternary eduction is the following analogue to that which tells us that, if A be an absolute binary reciprocant, $\left(\frac{1}{t^{\frac{1}{2}}}\frac{d}{dx}\right)A$ is another.

Let u, v be two absolute ternary reciprocants of characters κ, κ', so

that $$u = \omega^{\kappa}u' = \omega^{2\kappa}u'',$$

and $$v = \omega^{\kappa'}v' = \omega^{2\kappa'}v'',$$

then shall $\frac{1}{(pq)^{\frac{1}{3}}}\frac{d(u, v)}{d(x, y)}$ be an absolute reciprocant whose character is the residue (mod. 3) of $\kappa+\kappa'$.

We have at once

$$\frac{d(u, v)}{d(x, y)} = \omega^{\kappa+\kappa'}\left\{p\frac{du'}{dz}\left(\frac{dv'}{dy}+q\frac{dv'}{dz}\right)-p\frac{dv'}{dz}\left(\frac{du'}{dy}+q\frac{du'}{dz}\right)\right\}$$

$$= \omega^{\kappa+\kappa'}p\left\{\frac{du'}{dz}\frac{dv'}{dy}-\frac{dv'}{dz}\frac{du'}{dy}\right\}$$

$$= \omega^{\kappa+\kappa'}\left(\frac{pq}{p'q'}\right)^{\frac{1}{3}}\frac{d(u', v')}{d(y, z)}.$$

Thus, as was to be shown,

$$\frac{1}{(pq)^{\frac{1}{3}}}\frac{d(u, v)}{d(x, y)} = \omega^{\kappa+\kappa'}\frac{1}{(p'q')^{\frac{1}{3}}}\frac{d(u', v')}{d(y, z)} = \omega^{2(\kappa+\kappa')}\frac{1}{(p''q'')^{\frac{1}{3}}}\frac{d(u'', v'')}{d(z, x)} \quad \text{..............(34).}$$

An immediate consequence is that, if u, v, w, ϕ be four absolute ternary reciprocants, then

$$\frac{d(u, v)}{d(x, y)} \div \frac{d(w, \phi)}{d(x, y)} \quad \text{.........................(35)}$$

is another.

From (34) it follows that, if R and S be any two ternary reciprocants which become absolute upon division by $(pq)^m$ and $(pq)^n$ respectively, then

$$(R, S) = pq\frac{d(R, S)}{d(x, y)} - nS\frac{d(R, pq)}{d(x, y)} - mR\frac{d(pq, S)}{d(x, y)} \quad \text{......(36)}$$

is a reciprocant which becomes absolute upon division by $(pq)^{m+n+\frac{4}{3}}$, its character being the residue of the sum of the characters of R and S.

Let us, to avoid circumlocution, speak of m, the index of the power of pq which, dividing a reciprocant R, makes it absolute, as the *index of R*. Professor Sylvester uses the same word in the analogous sense in his theory.* Thus, R and S being of indices m and n respectively, that of the reciprocant here called (R, S) is $m+n+\frac{4}{3}$.

12. It will be well to consider for a moment some of the results obtained by (34) and (36), upon taking for u and v the absolute

reciprocants
$$\lambda = \log p + \omega \log q + \omega^2 \log(-1),$$
$$\mu = \log p + \omega^2 \log q + \omega \log(-1),$$
whose characters are 1 and 2 respectively.

The educed reciprocant (λ, μ), *i.e.*, $pq\dfrac{d(\lambda, \mu)}{d(x, y)}$, is at once

$$pq\begin{vmatrix} \dfrac{a_1}{p}+\omega\dfrac{b_1}{q}, & \dfrac{b_1}{p}+\omega\dfrac{c_1}{q} \\[2ex] \dfrac{a_1}{p}+\omega^2\dfrac{b_1}{q}, & \dfrac{b_1}{p}+\omega^2\dfrac{c_1}{q} \end{vmatrix},$$

or, omitting the factor $\omega^2-\omega$,

$$(\lambda, \mu) = a_1c_1 - b_1^2 \quad\text{..........................}(37),$$

which is the Hessian, of character zero and index $\frac{4}{3}$, as already seen by direct insertion.

From this we pass on to the higher educts, $\{\lambda, (\lambda, \mu)\}$ or (λ^2, μ), $\{(\lambda, \mu), \mu\}$ or (λ, μ^2), (λ^3, μ), (λ^2, μ^2), (λ, μ^3), &c., by successive application of (36). At each stage of the process only one of the second and third terms in (36) will appear, since at each either m or n is zero.

It is important to notice that

$$\frac{d(\lambda, pq)}{d(x, y)} = \begin{vmatrix} \dfrac{a_1}{p}+\omega\dfrac{b_1}{q}, & \dfrac{b_1}{p}+\omega\dfrac{c_1}{q} \\[2ex] qa_1+pb_1, & qb_1+pc_1 \end{vmatrix} = (1-\omega)(a_1c_1-b_1^2)\ldots(38),$$

and, similarly,
$$\frac{d(pq, \mu)}{d(x, y)} = (\omega^2-1)(a_1c_1-b_1^2) \quad\text{..................}(39).$$

Suppose, now, that $R = (\lambda^r, \mu^s)$ is any one of the series of educts,

* Referring to formulæ (4), we see that there are two forms of the equality expressive of the reciprocance of our R; viz., $(pq)^{-\frac{1}{3}\mu}R = (p'q')^{-\frac{1}{3}\mu}R'$, and $R = (-p)^{\mu}R'$. It would be in strict accordance with Professor Sylvester's nomenclature to speak of $\frac{\mu}{3}$ as the *index* of R when having the first form in mind, and of μ as the *characteristic* of R when regarding the second.

its character being κ and its index m. By (36), the two educts derived from this are, using (38) and (39),

$$(\lambda^{r+1}, \mu^{s}) = (\lambda, R) = pq\frac{d(\lambda, R)}{d(x, y)} - m(1-\omega)R(a_1c_1-b_1^2) \quad \ldots(40),$$

and $(\lambda^{r}, \mu^{s+1}) = (R, \mu) = pq\frac{d(R, \mu)}{d(x, y)} - m(\omega^2-1)R(a_1c_1-b_1^2) \quad \ldots(41),$

and are both of index $m+\frac{4}{3}$, the character of the first being $\kappa+1$ and that of the second $\kappa+2$, or the residues of those quantities. Now, $R(a_1c_1-b_1^2)$ is a reciprocant, of the same index $m+\frac{4}{3}$ as the two whole reciprocants of which it is a part, but is of character κ, *i.e.*, is of a different character from both of them. The second terms in (40) and (41) may not, therefore, be left out (as, from the fact of their being reciprocants, one might be tempted to assume) without vitiating the result. A simplification of this kind may, however, be introduced at any second stage in the development of educts as above. Thus, reapplying (41) to (40), we obtain for $(\lambda^{r+1}, \mu^{s+1})$ a somewhat complicated expression, of which the last term is

$$-m(3m+4)R(a_1c_1-b_1^2)^2,$$

i.e., is a reciprocant of the same index $m+\frac{8}{3}$ and character κ as the whole educt itself. The remaining terms by themselves therefore constitute such a reciprocant.

It may be worth while to write down the two educts (λ^2, μ) and (λ, μ^2). They are, omitting certain numerical multipliers,

$$(\lambda^2, \mu) = Q-\omega P \quad \ldots\ldots(42),$$

$$(\lambda, \mu^2) = Q-\omega^2 P \quad \ldots\ldots(43),$$

where $Q = 3q(a_1^2d_2-3a_1b_1c_2+a_1c_1b_2+2b_1^2b_2-b_1c_1a_2)-4(a_1c_1-b_1^2)^2$,

and $P = 3p(c_1^2a_2-3b_1c_1b_2+a_1c_1c_2+2b_1^2c_2-a_1b_1d_2)-4(a_1c_1-b_1^2)^2$,

their characters being 1 and 2 respectively, and the index of both $\frac{8}{3}$. From them may, of course, be derived the two reciprocants

$$Q^2+QP+P^2 \quad \ldots\ldots(44),$$

and $$(Q-P)(2Q+P)(Q+2P) \quad \ldots\ldots(45),$$

of character zero; and from these, respectively, the terms

$$16(a_1c_1-b_1^2)^4, \quad 128(a_1c_1-b_1^2)^6$$

may be omitted, as being, in each case, themselves reciprocants of the

same character and index as the reciprocants of which they are parts.

13. The following method will obtain no reciprocants which cannot be found by application of the results of § 11, but is of interest and may lead to important theory.

Let u, v, w be any three independent absolute ternary reciprocants, of characters $\kappa_1, \kappa_2, \kappa_3$ respectively. Let α, β, γ be any possible simultaneous increments of x, y, z; and let u_0, v_0, w_0 be the consequent increments of u, v, w; $u_0', v_0', w_0', u_0'', v_0'', w_0''$ those of $u', v', w', u'', v'', w''$ respectively. From the expressions of reciprocance of u, v, w, we have, upon subtraction of original from augmented values,

$$\left.\begin{aligned} u_0 &= \omega^{\kappa_1} u_0' = \omega^{2\kappa_1} u_0'' \\ v_0 &= \omega^{\kappa_2} v_0' = \omega^{2\kappa_2} v_0'' \\ w_0 &= \omega^{\kappa_3} w_0' = \omega^{2\kappa_3} w_0'' \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots(46).$$

Now, in virtue of our given relation between x, y, and z, it is theoretically possible, by elimination between this relation and the values found, upon differentiation and substitution, for u, v, w, to eliminate x, y, z, and express w as a function of u and v. We may, therefore, suppose the increment w_0 expanded in powers and products of powers of the increments u_0, v_0 by Taylor's theorem; and write

$$w_0 = \left(u_0 \frac{d}{du} + v_0 \frac{d}{dv}\right) w + \frac{1}{1.2}\left(u_0 \frac{d}{du} + v_0 \frac{d}{dv}\right)^2 w + \dots$$
$$\dots + \frac{1}{n!}\left(u_0 \frac{d}{du} + v_0 \frac{d}{dv}\right)^n w + \dots$$

Similarly,

$$w_0' = \left(u_0' \frac{d}{du'} + v_0' \frac{d}{dv'}\right) w' + \frac{1}{1.2}\left(u_0' \frac{d}{du'} + v_0' \frac{d}{dv'}\right)^2 w' + \dots$$
$$\dots + \frac{1}{n!}\left(u_0' \frac{d}{du'} + v_0' \frac{d}{dv'}\right)^n w' + \dots,$$

and $$w_0'' = \left(u_0'' \frac{d}{du''} + v_0'' \frac{d}{dv''}\right) w'' + \frac{1}{1.2}\left(u_0'' \frac{d}{du''} + v_0'' \frac{d}{dv''}\right) w'' + \dots$$
$$\dots + \frac{1}{n!}\left(u_0'' \frac{d}{du''} + v_0'' \frac{d}{dv''}\right)^n w'' + \dots$$

Multiplying, then, the second and third of these by ω^{κ_3} and $\omega^{2\kappa_3}$ respectively, and remembering the identities (46), we see that there are before us three apparently different expansions for the same quantity

w_0 in terms of the two independent quantities u_0, v_0. The three must be identical, and the various coefficients of powers and products of powers of u_0 and v_0 equal separately. Consequently, r and s being any positive integers whatever,

$$\frac{d^{r+s}w}{du^r dv^s} = \omega^{\kappa_3 - r\kappa_1 - s\kappa_2} \frac{d^{r+s}w'}{du'^r dv'^s} = \omega^{2(\kappa_3 - r\kappa_1 - s\kappa_2)} \frac{d^{r+s}w''}{du''^r dv''^s} \qquad (47).$$

In other words, r and s being any numbers, $\frac{d^{r+s}w}{du^r dv^s}$ is an absolute ternary reciprocant whose character is the residue (mod. 3) of

$$\kappa_3 - r\kappa_1 - s\kappa_2.$$

With a view to the actual calculation of these reciprocants, we must express the operators $\frac{d}{du}$ and $\frac{d}{dv}$, acting on a function of u and v, in terms of $\frac{d}{dx}$ and $\frac{d}{dy}$. Now, ϕ being any function of u and v, called Φ when transformed and expressed in terms of x and y,

$$\frac{d\phi}{du} = \frac{d(\phi, v)}{d(u, v)} = \frac{d(\Phi, v)}{d(x, y)} \div \frac{d(u, v)}{d(x, y)} \qquad (48),$$

and
$$\frac{d\phi}{dv} = \frac{d(u, \phi)}{d(u, v)} = \frac{d(u, \Phi)}{d(x, y)} \div \frac{d(u, v)}{d(x, y)} \qquad (49).$$

Thus the means of calculating the reciprocant $\frac{d^{r+s}w}{d^r u\, d^s v}$ is afforded.

14. Let us now fix our attention on one particular case of the above general theorem. Take for u, v, w the three linear absolute reciprocants, or sources of reciprocants, given in (1), (2), (3),

$$\zeta = z + x + y = \zeta' = \zeta'',$$

$$\xi = z + \omega x + \omega^2 y = \omega\xi' = \omega^2\xi'',$$

$$\eta = z + \omega^2 x + \omega y = \omega^2\eta' = \omega\eta''.$$

These are independent, since the determinant of the three linear expressions does not vanish. The reciprocants deduced as in (47) are then independent, and have for their type

$$\frac{d^{r+s}\zeta}{d\xi^r d\eta^s} = \omega^{2r+s} \frac{d^{r+s}\zeta'}{d\xi'^r d\eta'^s} = \omega^{r+2s} \frac{d^{r+s}\zeta''}{d\xi''^r d\eta''^s} \qquad (50).$$

It will now be proved that, in terms of these reciprocants, all absolute

reciprocants whatever, which do not involve x, y, or z explicitly, can be expressed.

Notice, first, that $\dfrac{d^{r+s}\zeta}{d\xi^r\,d\eta^s}$ involves $(r+s)^{\text{th}}$ differential coefficients of z with regard to x and y, and (it may be) all lower ones, but that it does not contain any higher differential coefficients, nor x, y, z explicitly. There are, then, $r+s+1$ absolute reciprocants of this series, which involve the $r+s+1$ elements a_{r+s}, b_{r+s}, c_{r+s}, ..., and lower, but no higher, derivatives of z with regard to x and y. On the whole, there are consequently $2+3+\ldots+m+1 = \frac{1}{2}(m^2+3m)$ reciprocants of the series involving no more than the $\frac{1}{2}(m^2+3m)$ earliest derivatives, whatever m be; and the $\frac{1}{2}(m^2+3m)$ expressions of their reciprocance give exactly the requisite number of relations necessary for the determination of p', q', a_1', b_1', c_1', ... a_m', b_m', c_m', ... in terms of p, q, a_1, b_1, c_1, ... a_m, b_m, c_m, ... Consequently, if there were another reciprocant, which could not be arrived at by composition of these, the expression of its reciprocance would give us the means of eliminating one set of derivatives entirely, and finding a relation in p, q, a_1, b_1, c_1 ... only. But these derivatives are independent. There is, therefore, no absolute reciprocant which cannot be expressed as desired.

It is an interesting conclusion, that the number of independent absolute ternary reciprocants involving elements up to and including any order is exactly the greatest number which could have been thought possible.

14. It is worth while to exemplify results (50) by calculating the three independent reciprocants linear in the second derivatives a_1, b_1, c_1. There are, of course, three such, viz., $\dfrac{d^2\zeta}{d\xi^2}$, $\dfrac{d^2\zeta}{d\xi\,d\eta}$, $\dfrac{d^2\zeta}{d\eta^2}$; but at present attention has only been called to one, viz., the "orthogonal" reciprocant (29).

It is easily verified that

$$\frac{d(\xi,\eta)}{d(x,y)},\quad \frac{d(\xi,\zeta)}{d(x,y)},\quad \frac{d(\zeta,\eta)}{d(x,y)}$$

are $(\omega-\omega^2)(p+q-1)$, $(1-\omega^2)(p+\omega q-\omega^2)$, $(\omega-1)(p+\omega^2 q-\omega)$,

respectively; so that, omitting numerical factors,

$$\frac{d\zeta}{d\xi} = \frac{p+\omega^2 q-\omega}{p+q-1} \quad\text{and}\quad \frac{d\zeta}{d\eta} = \frac{p+\omega q-\omega^2}{p+q-1} \quad \text{............(51).}$$

Hence, again omitting certain numerical factors, and remembering (6) that $p+q-1$ is a reciprocant of character zero and index $\frac{1}{3}$, we have, without difficulty, as three reciprocants,

$$(p+q-1)^3 \frac{d^2\zeta}{d\xi^2}, \quad (p+q-1)^3 \frac{d^2\zeta}{d\xi\, d\eta}, \quad (p+q-1)^3 \frac{d^2\zeta}{d\eta^2},$$

$$(q+\omega)^2 a_1 + (p+\omega^2)^2 c_1 - 2\,(p+\omega^2)(q+\omega)\, b_1 \quad \text{.........}(52),$$

$$(q^2-q+1)\, a_1 + (p^2-p+1)\, c_1 - \{2pq-p-q-1\}\, b_1 \quad \text{......}(53),$$

$$(q+\omega^2)^2 a_1 + (p+\omega)^2 c_1 - 2\,(p+\omega)(q+\omega^2)\, b_1 \text{...........}(54).$$

Each of the three is of index 1, and their characters are 1, 0, and 2 respectively. From them, and from two independent reciprocants involving p and q only, all other reciprocants involving no derivatives beyond the second can be derived by combination; for instance, (29) and (30). Or, adopting a reverse process, we may, from (53) and the numerator of (29), which are reciprocants of equal index and the same character, deduce what is probably the simplest ternary reciprocant linear in a_1, b_1, c_1; viz.,

$$qa_1 + pc_1 - (p+q+1)\, b_1 \text{ ....}(55).$$

Direct insertion of values from (26) here affords a verification.

15. The subject of orthogonal ternary reciprocants may be lightly touched upon. *Orthogonal absolute reciprocants* are such as remain unchanged by any transformation which in geometry of three dimensions expresses passage from one set of rectangular axes to another; and other reciprocants are orthogonal if they become orthogonal absolute reciprocants when made absolute by a power of $(1+p^2+q^2)$ as factor.

With the aid of geometrical knowledge, we have the means of writing down an infinite number of orthogonal reciprocants. By (35), if u, v, w, ϕ be four absolute reciprocants, then

$$\frac{d\,(u,\,v)}{d\,(x,\,y)} \div \frac{d\,(w,\,\phi)}{d\,(x,\,y)}$$

is another. Also, if u, v, w, ϕ be orthogonal, the determinant thus generated is also orthogonal; for, written in the form $\frac{d\,(u,\,v)}{d\,(w,\,\phi)}$, its expression introduces nothing depending on the particular rectangular axes.

Now, in (29) and (30) we have two absolute reciprocants, which are made orthogonal upon a simple preparation indicated at the end

of § 9. Take these, so prepared, for u and v. Moreover, in (25), which

tells us that $$\iint (1+p^2+q^2)^{\frac{1}{2}}\, dx\, dy$$

has the property of an absolute reciprocant, we possess more than is stated. For, in the first place it is orthogonal, being an expression for the area of the surface given by the relation between x, y, z; and, in the second place, its element $(1+p^2+q^2)^{\frac{1}{2}}\, dx\, dy$, *i.e.*, the element of surface, is equal to the product of the elements of any two functions whose Jacobian is $(1+p^2+q^2)^{\frac{1}{2}}$. Now, that element is the product of the arc elements ds, ds' of the lines of curvature through (x, y, z); and the expressions for these arcs s, s' can have no special reference to the particular axes. We know, then, two absolute orthogonal reciprocants s, s', which it is easier to interpret than to write down, and whose Jacobian is $(1+p^2+q^2)^{\frac{1}{2}}$. Let us take these for w and ϕ, whose Jacobian only is introduced above. We conclude that

$$\frac{1}{(1+p^2+q^2)^{\frac{1}{2}}} \cdot \frac{d(u, v)}{d(x, y)} \quad \text{.........................(56)},$$

where $$u = \frac{(1+q^2)\,a_1 - 2pqb_1 + (1+p^2)\,c_1}{(1+p^2+q^2)^{\frac{3}{2}}} \quad \text{..................(57)},$$

and $$v = \frac{a_1 c_1 - b_1^2}{(1+p^2+q^2)^2} \quad \text{..........................(58)}$$

is an orthogonal absolute reciprocant.

Writing U for (56), we now deduce two other absolute orthogonal

reciprocants, $$\frac{1}{(1+p^2+q^2)^{\frac{1}{2}}} \cdot \frac{d(U, v)}{d(x, y)}$$

and $$\frac{1}{(1+p^2+q^2)^{\frac{1}{2}}} \cdot \frac{d(u, U)}{d(x, y)};$$

and so, by repetition of the process, an infinite number.

It is to be expected that all orthogonal reciprocants may be derived by composition from those here obtained.

B. *n-ary Reciprocants.*

16. Many results of the foregoing part of this paper may be generalized. A brief elucidation of several of these generalisations follows; but no confidence is expressed that the best form is yet given to them.

Suppose there to be n variables $x_1, x_2, x_3, \ldots x_n$, connected by a single relation. Call by the names $p_1, p_2, \ldots p_{n-1}$ the first partial derivatives of x_n with regard to $x_1, x_2, \ldots x_{n-1}$; let $p_1', p_2', \ldots p_{n-1}'$ denote those of x_1, with regard to $x_2, x_3, \ldots x_n$, &c. &c. By $a_1, b_1, c_1, \ldots$, $a_1', b_1', c_1', \ldots, a_1'', b_1'', c_1'', \ldots$ will be meant, as before, the various second partial differential coefficients; by letters, other than x and p, with suffix 2, will be meant third partial differential coefficients; and similarly for suffixes 3, 4, &c. Unaccented letters will throughout treat x_n, singly accented x_1, doubly accented x_2, &c., as dependent variable; and the variables themselves will always be considered in the cyclical order of their suffixes.

Def. 1.—An *absolute n-ary reciprocant* is such a function of the partial derivatives $p_1, p_2, \ldots p_{n-1}, a_1, b_1, c_1, \ldots, a_2, b_2, c_2, \ldots$, or some of them (and it may be also of the variables themselves), as is equal, but for a constant factor to the same function of the derivatives $p_1', p_2', \ldots p_{n-1}', a_1', b_1', c_1', \ldots, a_2', b_2', c_2', \ldots$ (and of the variables each altered one stage in cyclical order in case of their explicit occurrence). The constant factor is always one of the n roots of $\rho^n - 1 = 0$.

Def. 2.—More generally an *n-ary reciprocant* is such a function as becomes an absolute n-ary reciprocant upon multiplication or division by some power of $(p_1 p_2 \ldots p_{n-1})^{1/n}$.

An absolute n-ary reciprocant, being unchanged in value, or only multiplied by a ρ, upon a cyclical substitution of the variables, is only again multiplied by that same ρ, upon a second such cyclical substitution; and so on for all n such successive substitutions. Similarly, the corresponding fact may be stated for non-absolute n-ary reciprocants.

Let $\rho_1, \rho_2, \rho_3, \ldots \rho_{n-1}, 1$ be the n roots of $\rho^n - 1 = 0$. There are n distinct kinds of ternary reciprocants, their characters depending on the root introduced in each case in the expression of reciprocance. There is, of course, a special similarity between the different characters in cases where n is a prime number, and in all cases those characters which correspond to primitive n^{th} roots of unity form a specially compact group.

The sources of n-ary reciprocants are the group of n linear functions of the variables themselves.

$$\left.\begin{aligned} &x_n + x_1 + x_2 + \ldots + x_{n-1} = \xi_n = \xi_n' = \xi_n'' = \ldots = \xi_n^{(n-1)} \\ &x_n + \rho_1 x_1 + \rho_1^2 x_2 + \ldots + \rho_1^{n-1} x_{n-1} = \xi_1 = \rho_1 \xi_1' = \rho_1^2 \xi_1'' = \ldots = \rho_1^{n-1} \xi_1^{(n-1)} \\ &x_n + \rho_{n-1} x_1 + \rho_{n-1}^2 x_2 + \ldots + \rho_{n-1}^{n-1} x_{n-1} \\ &\qquad = \xi_{n-1} = \rho_{n-1} \xi_{n-1}' = \rho_{n-1}^2 \xi_{n-1}'' = \ldots = \rho_{n-1}^{n-1} \xi_{n-1}^{(n-1)} \end{aligned}\right\} \ldots (59),$$

which may themselves, without causing confusion, be described as absolute reciprocants. One of them is of each character. By combination of these, or by immediate observation, all symmetric homogeneous functions of $x_1, x_2, \ldots x_n$ have the property of reciprocance.

17. The n equations connecting simultaneous infinitesimal variations of the variables, viz.,

$$dx_n = p_1 dx_1 + p_2 dx_2 + \ldots + p_{n-1} dx_{n-1},$$
$$dx_1 = p_1' dx_2 + p_2' dx_3 + \ldots + p_{n-1}' dx_n,$$
$$\&c. \qquad \&c.$$

are identical. The first two lead to the equalities,

$$\frac{-1}{p_{n-1}'} = \frac{p_1}{-1} = \frac{p_2}{p_1'} = \frac{p_3}{p_2'} = \ldots = \frac{p_{n-1}}{p_{n-2}'},$$

each of which is equal to $\left(\dfrac{p_1 p_2 \ldots p_{n-1}}{p_1' p_2' \ldots p_{n-1}'}\right)^{1/n}$ ....(60),

numerically (—but see next article).

Hence we conclude, as in § 5, that all homogeneous symmetric functions of $p_1, p_2, p_3, \ldots p_{n-1}, -1$ are n-ary reciprocants, and (an equivalent fact) that there are n independent linear n-ary reciprocants,

$$\left.\begin{array}{l} p_1 + p_2 + p_3 + \ldots + p_{n-1} - 1 \\ p_1 + \rho_1 p_2 + \rho_1^2 p_3 + \ldots + \rho_1^{n-2} p_{n-1} - \rho_1^{n-1} \\ p_1 + \rho_{n-1} p_2 + \rho_{n-1}^2 p_3 + \ldots + \rho_{n-1}^{n-2} p_{n-1} - \rho_{n-1}^{n-1} \end{array}\right\} \quad \ldots\ldots\ldots\ldots(61),$$

one of each of the different n characters.

Each of these linear reciprocants is of index $\frac{1}{n}$, *i.e.*, it is not absolute, but is made so upon division by $(p_1 p_2 \ldots p_{n-1})^{1/n}$. The symmetric homogeneous function of the r^{th} degree

$$S_r(p_1, p_2, \ldots p_{n-1}, -1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(62)$$

is of index $\frac{r}{n}$. In particular, the reciprocant

$$1 + p_1^2 + p_2^2 + \ldots + p_{n-1}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(63),$$

which is no doubt connected with interesting propositions best expressed in the language of geometry of n dimensions, is of index $\frac{2}{n}$.

From equations (60) we also, just as in § 6, draw the conclusion that the $n-1$ linear expressions in logarithms

$$\left.\begin{array}{l} \log p_1 + \rho_1 \log p_2 + \rho_1^2 \log p_3 + \ldots + \rho_1^{n-2} \log p_{n-1} + \rho_1^{n-1} \log(-1) \\ \log p_1 + \rho_2 \log p_2 + \rho_2^2 \log p_3 + \ldots + \rho_2^{n-2} \log p_{n-1} + \rho_2^{n-1} \log(-1) \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \log p_1 + \rho_{n-1} \log p_2 + \rho_{n-1}^2 \log p_3 + \ldots + \rho_{n-1}^{n-2} \log p_{n-1} + \rho_{n-1}^{n-1} \log(-1) \end{array}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (64)$$

are absolute n-ary reciprocants, each of the character corresponding to the root occurring in its expression.

18. There is a little difficulty as to the extraction of the n^{th} root in (60). It would almost appear that, for some reason imperfectly understood, that root should be extracted negatively when n is even, as is certainly the case when $n=2$, and as is suggested by the following evaluation of the Jacobian of $x_1, x_2, \ldots x_{n-1}$ expressed in terms of $x_2, x_3, \ldots x_n$,

$$\frac{d(x_1, x_2, \ldots x_{n-1})}{d(x_2, x_3, \ldots x_n)} = \begin{vmatrix} p_1', & p_2', & \ldots\ldots & p_{n-1}' \\ 1, & 0, & \ldots\ldots & 0 \\ 0, & 1, & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots\ldots & 1 \end{vmatrix} = (-1)^n p_{n-1}'$$

$$= (-1)^{n-1} \left(\frac{p_1' p_2' \ldots p_{n-1}'}{p_1 p_2 \ldots p_{n-1}}\right)^{1/n} \ldots\ldots\ldots\ldots\ldots (65),$$

the root being extracted with the same sign character as in (60). Now the cases of $n=2$ and $n=3$ certainly indicate that this sign character when multiplied by $(-1)^{n-1}$ should be positive unity.

However this be, we can draw from the undoubted part of our present conclusion the fact that

$$\iiint \ldots (p_1 p_2 \ldots p_{n-1})^{1/n}\, dx_1\, dx_2 \ldots dx_n \quad \ldots\ldots\ldots\ldots\ldots (66)$$

has the property of an absolute reciprocant, the only question being whether it is always of character corresponding to the root unity, or of that corresponding to minus unity when n is even and greater than 2.

Hence, also, R being an n-ary reciprocant of index $\frac{1}{n}$,

$$\iiint \dots R\, dx_1\, dx_2 \dots dx_n$$

has the property of an absolute reciprocant.

19. Proceeding now to the subject of the eduction of n-ary reciprocants from others, we can see that, if $u_1, u_2, \dots u_{n-1}$ be any $n-1$ absolute n-ary reciprocants, the Jacobian

$$J = \frac{d(u_1, u_2, \dots u_{n-1})}{d(x_1, x_2, \dots x_{n-1})}$$

is an n-ary reciprocant of index $\frac{1}{n}$.

Proceeding, as in § 11, we obtain, supposing u to be of character corresponding to the root of unity ρ,

$$\left.\begin{aligned}
\frac{du}{dx_1} &= \rho p_1 \frac{du'}{dx_n} \\
\frac{du}{dx_2} &= \rho \left\{ \frac{du'}{dx_2} + p_2 \frac{du'}{dx_n} \right\} \\
\frac{du}{dx_3} &= \rho \left\{ \frac{du'}{dx_3} + p_3 \frac{du}{dx_n} \right\} \\
\dots & \quad \dots \quad \dots \quad \dots \quad \dots \\
\frac{du}{dx_{n-1}} &= \rho \left\{ \frac{du'}{dx_{n-1}} + p_{n-1} \frac{du'}{dx_n} \right\}
\end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (67).$$

Hence, denoting by $\Pi\rho$ the product of the roots of unity which determine the characters of $u_1, u_2, \dots u_{n-1}$, we obtain, upon insertion in J,

$$J = \Pi\rho \left\{ (-1)^n p_1 J' + \text{a sum of determinants with two rows identical} \right\}$$

$$= \Pi\rho \left(\frac{p_1 p_2 \dots p_n}{p_1' p_2' \dots p_{n-1}'} \right)^{1/n} J',$$

subject to the reservation, as to the particular n^{th} root intended, alluded to above.

Now $\Pi\rho$ is a root of $\rho^n - 1 = 0$. Call it ρ', then

$$\frac{J}{(p_1 p_2 \dots p_{n-1})^{1/n}} = \rho' \frac{J'}{(p_1' p_2' \dots p_{n-1}')^{1/n}} = \rho'^2 \frac{J''}{(p_1'' p_2'' \dots p_{n-1}'')^{1/n}} = \dots \quad \dots\dots\dots\dots (68),$$

i.e., J is an n-ary reciprocant, as stated.

If we apply this proposition, taking for $u_1, u_2, \dots u_{n-1}$ the $n-1$ linear logarithmic absolute reciprocants (64), we obtain readily that

$$\begin{vmatrix} a_1, & b_1, & c_1, & \dots \\ b_1, & c_1, & d_1, & \dots \\ c_1, & d_1, & e_1, & \dots \\ \dots & \dots & \dots \end{vmatrix} \times \begin{vmatrix} \frac{1}{p_1}, & \frac{\rho_1}{p_2}, & \frac{\rho_1^2}{p_3}, & \dots \\ \frac{1}{p_1}, & \frac{\rho_2}{p_2}, & \frac{\rho_2^2}{p_3}, & \dots \\ \frac{1}{p_1}, & \frac{\rho_3}{p_2}, & \frac{\rho_3^2}{p_3}, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

is an n-ary reciprocant of index $\frac{1}{n}$. It follows that the Hessian

$$\begin{vmatrix} a_1, & b_1, & c_1, & \dots \\ b_1, & c_1, & d_1, & \dots \\ c_1, & d_1, & e_1, & \dots \\ \dots & \dots & \dots \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (69)$$

is one of index $\frac{n+1}{n}$. This is the earliest instance of a pure n-ary reciprocant.

Another Jacobian theorem, a generalisation of (22), is that since, as it is easy to verify,

$$\frac{d(p_1, p_2, \dots p_{n-1})}{d(p_1', p_2', \dots p_{n-1}')} = \frac{p_1 p_2 \dots p_{n-1}}{p_1' p_2' \dots p_{n-1}'},$$

the Jacobian of $n-1$ absolute n-ary reciprocants, involving $p_1, p_2, \dots p_{n-1}$ only, with regard to $p_1, p_2, \dots p_{n-1}$, is itself a reciprocant of index -1.

20. Again, as in § 13, we have the general theorem, that

$$u_1, u_2, u_3, \dots u_{n-1}, u_n,$$

being n independent absolute n-ary reciprocants, any such derivative

as
$$\frac{d^{r_1+r_2+\dots+r_{n-1}} u_n}{du_1^{r_1} du_2^{r_2} \dots du_{n-1}^{r_{n-1}}} \quad \dots\dots\dots\dots\dots\dots\dots\dots (70)$$

is an n-ary reciprocant, also absolute, and of character determined by the factor $\rho_n \rho_1^{-r_1} \rho_2^{-r_2} \dots \rho_{n-1}^{-r_{n-1}}$.

Such absolute reciprocants are to be calculated by aid of the

Hence, also, R being an n-ary reciprocant of index $\frac{1}{n}$,

$$\iiint \dots R\, dx_1\, dx_2 \dots dx_n$$

has the property of an absolute reciprocant.

19. Proceeding now to the subject of the eduction of n-ary reciprocants from others, we can see that, if $u_1, u_2, \dots u_{n-1}$ be any $n-1$ absolute n-ary reciprocants, the Jacobian

$$J = \frac{d\,(u_1, u_2, \dots u_{n-1})}{d\,(x_1, x_2, \dots x_{n-1})}$$

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$$\left.\begin{aligned} \frac{du}{dx_1} &= \rho p_1 \frac{du'}{dx_n} \\ \frac{du}{dx_2} &= \rho \left\{ \frac{du'}{dx_2} + p_2 \frac{du'}{dx_n} \right\} \\ \frac{du}{dx_3} &= \rho \left\{ \frac{du'}{dx_3} + p_3 \frac{du}{dx_n} \right\} \\ \dots & \quad \dots \quad \dots \quad \dots \quad \dots \\ \frac{du}{dx_{n-1}} &= \rho \left\{ \frac{du'}{dx_{n-1}} + p_{n-1} \frac{du'}{dx_n} \right\} \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (67).$$

Hence, denoting by $\Pi\rho$ the product of the roots of unity which determine the characters of $u_1, u_2, \dots u_{n-1}$, we obtain, upon insertion in J,

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$$= \Pi\rho \left(\frac{p_1 p_2 \dots p_n}{p_1' p_2' \dots p_{n-1}'} \right)^{1/n} J',$$

subject to the reservation, as to the particular n^{th} root intended, alluded to above.

Now $\Pi\rho$ is a root of $\rho^n - 1 = 0$. Call it ρ', then

$$\frac{J}{(p_1 p_2 \dots p_{n-1})^{1/n}} = \rho' \frac{J'}{(p_1' p_2' \dots p_{n-1}')^{1/n}} = \rho'^2 \frac{J''}{(p_1'' p_2'' \dots p_{n-1}'')^{1/n}} = \dots \quad \dots\dots\dots\dots (68),$$

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$$\begin{vmatrix} a_1, & b_1, & c_1, & \dots \\ b_1, & c_1, & d_1, & \dots \\ c_1, & d_1, & e_1, & \dots \\ \dots & \dots & \dots & \end{vmatrix} \times \begin{vmatrix} \frac{1}{p_1}, & \frac{\rho_1}{p_2}, & \frac{\rho_1^2}{p_3}, & \dots \\ \frac{1}{p_1}, & \frac{\rho_2}{p_2}, & \frac{\rho_2^2}{p_3}, & \dots \\ \frac{1}{p_1}, & \frac{\rho_3}{p_2}, & \frac{\rho_3^2}{p_3}, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

is an n-ary reciprocant of index $\frac{1}{n}$. It follows that the Hessian

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the Jacobian of $n-1$ absolute n-ary reciprocants, involving $p_1, p_2, \dots p_{n-1}$ only, with regard to $p_1, p_2, \dots p_{n-1}$, is itself a reciprocant of index -1.

20. Again, as in § 13, we have the general theorem, that

$$u_1, u_2, u_3, \dots u_{n-1}, u_n,$$

being n independent absolute n-ary reciprocants, any such derivative

as $$\frac{d^{r_1+r_2+\dots+r_{n-1}} u_n}{du_1^{r_1} du_2^{r_2} \dots du_{n-1}^{r_{n-1}}} \quad \dots\dots\dots\dots\dots\dots (70)$$

is an n-ary reciprocant, also absolute, and of character determined by the factor $\rho_n \rho_1^{-r_1} \rho_2^{-r_2} \dots \rho_{n-1}^{-r_{n-1}}$.

Such absolute reciprocants are to be calculated by aid of the

Hence, also, R being an n-ary reciprocant of index $\dfrac{1}{n}$,

$$\iiint \dots R\, dx_1\, dx_2 \dots dx_n$$

has the property of an absolute reciprocant.

19. Proceeding now to the subject of the eduction of n-ary reciprocants from others, we can see that, if $u_1, u_2, \dots u_{n-1}$ be any $n-1$ absolute n-ary reciprocants, the Jacobian

$$J = \frac{d\,(u_1, u_2, \dots u_{n-1})}{d\,(x_1, x_2, \dots x_{n-1})}$$

is an n-ary reciprocant of index $\dfrac{1}{n}$.

Proceeding, as in § 11, we obtain, supposing u to be of character corresponding to the root of unity ρ,

$$\left.\begin{aligned} \frac{du}{dx_1} &= \rho p_1 \frac{du'}{dx_n} \\ \frac{du}{dx_2} &= \rho \left\{ \frac{du'}{dx_2} + p_2 \frac{du'}{dx_n} \right\} \\ \frac{du}{dx_3} &= \rho \left\{ \frac{du'}{dx_3} + p_3 \frac{du}{dx_n} \right\} \\ \dots & \quad \dots \quad \dots \quad \dots \quad \dots \\ \frac{du}{dx_{n-1}} &= \rho \left\{ \frac{du'}{dx_{n-1}} + p_{n-1} \frac{du'}{dx_n} \right\} \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (67).$$

Hence, denoting by $\Pi\rho$ the product of the roots of unity which determine the characters of $u_1, u_2, \dots u_{n-1}$, we obtain, upon insertion in J,

$$\begin{aligned} J &= \Pi\rho \left\{ (-1)^n p_1 J' + \text{a sum of determinants with two rows identical} \right\} \\ &= \Pi\rho \left(\frac{p_1 p_2 \dots p_n}{p_1' p_2' \dots p_{n-1}'} \right)^{1/n} J', \end{aligned}$$

subject to the reservation, as to the particular n^{th} root intended, alluded to above.

Now $\Pi\rho$ is a root of $\rho^n - 1 = 0$. Call it ρ', then

$$\frac{J}{(p_1 p_2 \dots p_{n-1})^{1/n}} = \rho' \frac{J'}{(p_1' p_2' \dots p_{n-1}')^{1/n}} = \rho'^2 \frac{J''}{(p_1'' p_2'' \dots p_{n-1}'')^{1/n}} = \dots \quad \dots\dots\dots\dots (68),$$

i.e., J is an n-ary reciprocant, as stated.

If we apply this proposition, taking for $u_1, u_2, \ldots u_{n-1}$ the $n-1$ linear logarithmic absolute reciprocants (64), we obtain readily that

$$\begin{vmatrix} a_1, & b_1, & c_1, & \ldots \\ b_1, & c_1, & d_1, & \ldots \\ c_1, & d_1, & e_1, & \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix} \times \begin{vmatrix} \frac{1}{p_1}, & \frac{\rho_1}{p_2}, & \frac{\rho_1^2}{p_3}, & \ldots \\ \frac{1}{p_1}, & \frac{\rho_2}{p_2}, & \frac{\rho_2^2}{p_3}, & \ldots \\ \frac{1}{p_1}, & \frac{\rho_3}{p_2}, & \frac{\rho_3^2}{p_3}, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

is an n-ary reciprocant of index $\frac{1}{n}$. It follows that the Hessian

$$\begin{vmatrix} a_1, & b_1, & c_1, & \ldots \\ b_1, & c_1, & d_1, & \ldots \\ c_1, & d_1, & e_1, & \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(69)$$

is one of index $\frac{n+1}{n}$. This is the earliest instance of a pure n-ary reciprocant.

Another Jacobian theorem, a generalisation of (22), is that since, as it is easy to verify,

$$\frac{d\,(p_1, p_2, \ldots p_{n-1})}{d\,(p_1', p_2', \ldots p_{n-1}')} = \frac{p_1 p_2 \ldots p_{n-1}}{p_1' p_2' \ldots p_{n-1}'},$$

the Jacobian of $n-1$ absolute n-ary reciprocants, involving $p_1, p_2, \ldots p_{n-1}$ only, with regard to $p_1, p_2, \ldots p_{n-1}$, is itself a reciprocant of index -1.

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$$u_1, u_2, u_3, \ldots u_{n-1}, u_n,$$

being n independent absolute n-ary reciprocants, any such derivative

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is an n-ary reciprocant, also absolute, and of character determined by the factor $\rho_n \rho_1^{-r_1} \rho_2^{-r_2} \ldots \rho_{n-1}^{-r_{n-1}}$.

Such absolute reciprocants are to be calculated by aid of the

Jacobian theorems

$$\frac{d\phi}{du_1} = \frac{d(\phi, u_2, u_3, \dots u_{n-1})}{d(u_1, u_2, u_3, \dots u_{n-1})}$$

$$= \frac{d(\phi, u_2, u_3, \dots u_{n-1})}{d(x_1, x_2, x_3, \dots x_{n-1})} \div \frac{d(u_1, u_2, u_3, \dots u_{n-1})}{d(x_1, x_2, x_3, \dots x_{n-1})},$$

$$\frac{d\phi}{du_2} = \frac{d(u_1, \phi, u_3, \dots u_{n-1})}{d(x_1, x_2, x_3, \dots x_{n-1})} \div \frac{d(u_1, u_2, u_3, \dots u_{n-1})}{d(x_1, x_2, x_3, \dots x_{n-1})}, \text{ \&c., \&c.}$$

Finally, as in § 14, it may be seen that the whole subject of n-ary reciprocants is in reality enshrined in the n linear sets of reciprocantive identities (59); and that the number of independent n-ary reciprocants involving partial differential coefficients not higher than the m^{th}, and free from the variables explicitly, is exactly the number of those differential coefficients, that is to say,

$$\frac{(m+n-1)!}{m!\,(n-1)!} - 1.$$

Thursday, April 8th, 1886.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

The following communications were made:—

On the Number of linearly independent Invariants (or Seminvariants), Reciprocants, or in general of Integrals of any assigned type of a homogeneous and isobaric linear Partial Differential Equation: Prof. Sylvester, F.R.S.

On some Results connected with the Theory of Reciprocants: C. Leudesdorf, M.A.

The President (Mr. Walker, F.R.S., Vice-President, in the Chair) gave an account of the work he has been for some time engaged upon in connection with Elliptic Functions, the special points he drew attention to being the use of the *twelve* Elliptic Functions and of twelve Zeta and twelve Theta Functions. The two latter systems of functions depend upon the quantities E, G, I, where $G = E - k'K$, and $I = E - K$.

Mr. Kempe, F.R.S., next communicated a Note on an Extension of

ordinary Algebra, differing from the latter in the substitution of three arbitrary quantities z, i, and u for the quantities 0, 1, and ∞.

Mr. Tucker read a Note, A Theorem in Conics, by the Rev. T. C. Simmons, M.A.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xxxix., No. 241.

"Proceedings of the Cambridge Philosophical Society," Vol. v., Part 5, Mich. 1885.

"Mathematical Questions, with their Solutions, from the 'Educational Times,'" Vol. xliv.

"Educational Times," April, 1886.

"Proceedings of the Canadian Institute," Third Series, Vol. iii., Fasc. No. 3; Toronto, 1886.

"Jahrbuch über die Fortschritte der Mathematik," xv., 2, Jahrgang 1883.

"Bulletin des Sciences Mathématiques," T. x., March and April, 1886.

"Bulletin de la Société Mathématique de France," T. xiv., No. 1.

"Beiblätter zu den Annalen der Physik und Chemie," B. x., St. 3, 1886.

"Catalogue de la Bibliothèque de l'Ecole Polytechnique," 8vo; Paris, 1881.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. ii., F. 4, 5, 6, Feb., March, 1886.

"Atti del R. Istituto Veneto," T. ii., Ser. v., Disp. 3 to 10; T. iii., Ser. vi., Disp. 1 to 9; 1883–85.

"Memorie del R. Istituto Veneto," Vol. xxii., Parts i. and ii.; di Scienze, Lettere, ed Arti, 1884–85.

On some Results connected with the Theory of Reciprocants.

By C. Leudesdorf, M.A.

[*Read April 8th*, 1886.]

1. Let x and y be two variables connected by any relation, and let $y_1, y_2, \ldots$ denote the successive differential coefficients of y with respect to x, and $x_1, x_2, \ldots$ those of x with respect to y. Then

$$\begin{aligned}
x_1 &= 1 && \div y_1,\\
x_2 &= -y_2 && \div y_1^3,\\
x_3 &= -y_1y_3+3y_2^2 && \div y_1^5,\\
x_4 &= -y_1^2y_4+10y_1y_2y_3-15y_2^3 && \div y_1^7,
\end{aligned}$$

and so on. If the numerators on the right be denoted by Y_1, Y_2, ..., we notice the following properties of Y_n :—

It is homogeneous, and every term of it is of degree $n-1$.

It is isobaric, and every term of it is of weight $2(n-1)$ (if the weight of y_r be taken to be r).

It only involves y_n once, in the term $-y_1^{n-2}y_n$.

The denominator corresponding to it is y_1^{2n-1}.

2. Let $R(y_1, y_2, \ldots)$ be a reciprocant every term of which is of degree i and weight w, and which is equal to $qy_1^{\lambda}R(x_1, x_2, \ldots)$, where $q = \pm 1$. Then evidently q is $+1$ or -1 according as the number of factors in that term of R which contains the highest power of y_1 is (neglecting the power of y_1) even or odd. For

$$y_1 = +\frac{x_1}{x_1^2},$$

but
$$y_2 = -\frac{x_2}{x_1^3}, \quad y_3 = -\frac{x_3}{x_1^4} + \ldots,$$

and in general
$$y_r = -\frac{x_r}{x_1^{r+1}} + \ldots.$$

If R does not contain y_1, then evidently $q = (-1)^i$.

Again, λ must be equal to the sum of the degree (i) and the weight (w) of R. For, if we write ky for y (where k is a constant), y_r^m becomes $k^m y_r^m$, but x_r^m becomes $k^{-mr}x_r^m$; thus every term of $R(y_1, y_2, \ldots)$ will be multiplied by k^i and every term of $R(x_1, x_2, \ldots)$ will be multiplied by k^{-w}; therefore

$$k^i R(y_1, y_2, \ldots) = qk^{\lambda}y_1^{\lambda}k^{-w}R(x_1, x_2, \ldots),$$

so that $k^i = k^{\lambda-w}$, which shows that $\lambda = w+i$. More generally, it is seen that any homogeneous and isobaric function $F(y_1, y_2, \ldots)$ of degree i and weight w will, if transformed by means of the formulæ reciprocal to those in § 1, become a homogeneous and isobaric function $y_1^{w+i}\Phi(x_1, x_2, \ldots)$ of $x_1, x_2, \ldots$.

3. Let now F be *any* function of $y_1, y_2, \ldots$; it can of course be expressed in terms of $x_1, x_2, \ldots$; let it become $x_1^{-\lambda}\Phi(x_1, x_2, \ldots)$ when so expressed. If we write $x - y\theta$ (where θ is an infinitesimal) in place of x, and leave y unaltered, the change in Φ will be

$$-\theta\frac{d}{dx_1}(\Phi x_1^{-\lambda});$$

for the change makes x_1 into $x_1-\theta$, and x_2, x_3, ... are unaffected by it. Let us examine the effect of the change upon F.

We have $\delta y = 0$, $\delta x = -y\theta$, and therefore, writing $\omega = \delta y - y_1 \delta x = yy_1\theta$,

$$\delta y_1 = \omega' - yy_2\theta = y_1^2\theta,$$

$$\delta y_2 = \omega'' - yy_3\theta = 3y_1y_2\theta,$$

$$\delta y_3 = \omega''' - yy_4\theta = (4y_1y_3+3y_2^2)\,\theta,$$

and so on. Therefore

$$\delta F = \frac{dF}{dy_1}\delta y_1 + \frac{dF}{dy_2}\delta y_2 + \frac{dF}{dy_3}\delta y_3 + \dots$$

$$= \left[y_1^2\frac{d}{dy_1} + 3y_1y_2\frac{d}{dy_2} + (4y_1y_3+3y_2^2)\frac{d}{dy_3} + \dots\right]F\theta$$

$$= \left[-y_1^2\frac{d}{dy_1} + y_1\left(2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + \dots\right) + V\right]F\theta,$$

where V is the well-known operator

$$3y_2^2\frac{d}{dy_3} + 10y_2y_3\frac{d}{dy_4} + \dots,$$

the coefficient of $\dfrac{d}{dy_m}$ in V being

$$\frac{d^m(yy_1)}{dx^m} - yy_{m+1} - (m+1)\,y_1y_m,$$

that is,
$$c_2^m y_2y_{m-1} + c_3^m y_3y_{m-2} + \dots + c_{m-1}^m y_{m-1}y_2,$$

c_r^m denoting the number of combinations of m things taken r together. Equating the changes in F and Φ, there results

$$y_1^2\frac{dF}{dy_1} - y_1\left(2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + \dots\right)F - VF = \frac{d\,(\Phi x_1^{-\lambda})}{dx_1} \quad \dots\dots(1).$$

In the case where F is homogeneous and isobaric, this last equation can be simplified. For, if F be of degree i and weight w, then, by Euler's theorem and the isobaric theorem,

$$\left(y_1\frac{d}{dy_1} + y_2\frac{d}{dy_2} + \dots\right)F = iF,$$

$$\left(y_1\frac{d}{dy_1} + 2y_2\frac{d}{dy_2} + \dots\right)F = wF,$$

so that the equation reduces in this case to

$$y_1^2 \frac{dF}{dy_1} - (w+i)\, y_1 F - VF = \frac{d}{dx_1}\,(\Phi x_1^{-\lambda}),$$

or, since $\lambda = w+i$ (§ 2), to

$$y_1^2 \frac{dF}{dy_1} - VF = y_1^{(w+i)} \frac{d\Phi}{dx_1} \quad \text{......................(2)}.$$

If F is a *pure* function, *i.e.*, one which does not contain y_1, then

$$VF = -y_1^{w+i} \frac{d\Phi}{dx_1} \text{...........................(3)},$$

a result in which is included the well-known proposition that if F is a *pure reciprocant* $\qquad VF = 0.$

4. The following preliminary proposition will be required in § 5.

Let $F(y_2, y_3, \ldots)$ be a rational homogeneous isobaric function, of degree i and weight w, of the differential coefficients of y with respect to x (*excluding the first*). If $y_1^{-(w+i)} F$ is such that it is unchanged by the substitution of $x - y\theta$ for x (where θ is an infinitesimal), y remaining unaltered, then F must be a pure reciprocant.

To prove this, let the substitutions

$$(1)\ \left.\begin{aligned} x &= X - Y \\ y &= Y \end{aligned}\right\}, \qquad (2)\ \left.\begin{aligned} X &= X' \\ Y &= X' + Y' \end{aligned}\right\}, \qquad (3)\ \left.\begin{aligned} X' &= \xi - \eta \\ Y' &= \eta \end{aligned}\right\}$$

be made successively in $y_1^{-(w+i)} F$. Since, by hypothesis, this function is unchanged when $x - y\theta$, y are written for x, y respectively, it follows that any number of such infinitesimal changes made successively in x will have no effect on the function, and therefore that we may write $x - y$ for x and y for y without altering it. Therefore

$$y_1^{-(w+i)} F(y_2, y_3, \ldots) = Y_1^{-(w+i)} F(Y_2, Y_3, \ldots) \text{ ....(4)},$$

where Y_r denotes $\dfrac{d^r Y}{dX^r}$.

Now let the second substitution be made in the right-hand member of (4). We have

$$\frac{dY}{dX} = 1 + \frac{dY'}{dX'};$$

but

$$\frac{d^2 Y}{dX^2} = \frac{d^2 Y'}{dX'^2}, \ \&c.,$$

and so in general $Y_r = Y'_r$ except when $r = 1$. Accordingly

$$Y_1^{-(w+i)} F(Y_2, Y_3, \ldots) = (1+Y'_1)^{-(w+i)} F(Y'_2, Y'_3, \ldots)$$

$$= Y_1'^{-(w+i)} F(Y'_2, Y'_3, \ldots) \left(\frac{Y'_1}{1+Y'_1}\right)^{w+i} \ldots\ldots (5).$$

Now let the third substitution be made in the right-hand member of (5); it will evidently become

$$\eta_1^{-(w+i)} F(\eta_2, \eta_3, \ldots)\, \eta_1^{w+i}$$

$\left(\text{where } \eta_r \text{ denotes } \dfrac{d^r\eta}{d\xi^r}\right)$,

that is, $F(\eta_2, \eta_3, \ldots)$ simply.

But now $\qquad x = X - Y = X' - (X'+Y') = -\eta,$

$\qquad\qquad\quad y = Y \quad = X'+Y' \qquad = \xi,$

so that the effect of the train of three substitutions is to change x into $-y$, and y into x. And, since $\eta_2 = -x_2$, $\eta_3 = -x_3$, &c., therefore $F(\eta_2, \eta_3, \ldots)$ is equal to $(-1)^i F(x_2, x_3, \ldots)$. It has therefore been shown

that $\qquad y_1^{-(w+i)} F(y_2, y_3, \ldots) = (-1)^i F(x_2, x_3, \ldots);$

i.e., F is a pure reciprocant.

5. The results of the preceding articles may now be made use of to prove the converse of the proposition mentioned at the end of § 3; viz., that if F is a rational integral homogeneous isobaric function of $y_2, y_3, \ldots$, then, if $VF = 0$, F must be a pure reciprocant.

Let x be changed into $x - y\theta$, as in § 3; then, as already seen, F is changed to $F + \delta F$, where

$$\delta F = \left\{-y_1^2 \frac{dF}{dy_1} + (w+i)\, F + VF\right\} \theta = (w+i)\, F\theta,$$

the other terms vanishing by hypothesis. Since $\delta y_1 = y_1^2\theta$, this can

be written $\qquad \delta\left\{y_1^{-(w+i)} F\right\} = 0,$*

which shows (§ 4) that F is a pure reciprocant.

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* This result may also be seen from equation (3) of § 3, which shows that, if $F(y_2, y_3, \ldots)$ be transformed by substituting for $y_2, y_3, \ldots$ their values in terms of $x_1, x_2, x_3, \ldots$, and become $x_1^{-(w+i)} \Phi(x_1, x_2, x_3, \ldots)$, then, when $VF = 0$, also $\dfrac{d\Phi}{dx_1} = 0$; that is, Φ does not involve x_1. Accordingly

$$\delta\left\{y_1^{-(w+i)} F\right\} = \delta\Phi(x_2, x_3, \ldots) = 0.$$

6. Let F now stand again for *any* function of $y_1, y_2, \ldots$; and let an infinitesimal orthogonal change be given to x and y; *i.e.*, let x become $x-y\theta$ and y become $y+x\theta$, where θ is infinitesimal. Then proceeding as in § 3 to find the change in F, we have

$$\omega = \delta y - y_1\,\delta x = (x+yy_1)\,\theta,$$

therefore $$\delta y_1 = (1+y_1^2)\,\theta,$$

and δy_2, δy_3, &c. have the same values as given in § 3. Thus

$$\delta F = \left\{ (1+y_1^2)\frac{dF}{dy_1} + y_1\left(3y_2\frac{dF}{dy_2} + 4y_3\frac{dF}{dy_3} + \ldots\right) + VF \right\}\theta \;\ldots\ldots(6).$$

In the case, then, where F is an absolute orthogonal reciprocant O,

$$(1-y_1^2)\frac{dO}{dy_1} + y_1\left(2y_1\frac{dO}{dy_1} + 3y_2\frac{dO}{dy_2} + \ldots\right) + VO = 0 \quad \ldots\ldots(7),$$

or, say, $$U\,.\,O = 0.$$

If O be an orthogonal reciprocant, but no longer an absolute one, then we can make it into an absolute one by dividing it by a suitable power of y_2; if this power be the k^{th}, then

$$U\,.\,O = 3ky_1 O \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

For
$$\begin{aligned} U\,(Oy_2^{-k}) &= y_2^{-k}UO - ky_2^{-(k+1)}O\delta y_2\theta^{-1} \\ &= y_2^{-k}UO - 3ky_1y_2^{-k}O \\ &= y_2^{-k}\,(UO - 3ky_1O) \;\ldots\ldots\ldots\ldots\ldots\ldots(9). \end{aligned}$$

7. If F is a function of $y_1, y_2, \ldots$ such that

$$UF = \mu y_1 F,$$

where μ is some constant, then F must be a reciprocant, and an orthogonal one; such, moreover, that $y_2^{-\frac{1}{3}\mu}F$ is an *absolute* orthogonal reciprocant.

This proposition, the converse of that given in equation (7) of § 6, is easily proved. For, if $x-y\theta$, $y+x\theta$ be written for x and y, as in § 6, the change made in $y_2^{-\frac{1}{3}\mu}F$

$$\begin{aligned} &= U\,(y_2^{-\frac{1}{3}\mu}F) && \text{by (7)} \\ &= y_2^{-\frac{1}{3}\mu}\,(UF - \mu y_1 F) && \text{by (9)} \\ &= 0, \end{aligned}$$

by hypothesis.

Therefore $y_2^{-\frac{1}{3}\mu}F$ is not altered by an infinitesimal orthogonal change

given to x and y; and therefore is not altered by any number of such changes made successively; that is to say, by any orthogonal change in the variables. In other words, it is an absolute orthogonal reciprocant.

8. Let $R(y_1, y_2, \dots)$ be any reciprocant; let it be made absolute by division by a suitable power of y_2, say the k^{th}. Thus,

$$y_2^{-k}R(y_1, y_2, \dots) = \pm x_2^{-k}R(x_1, x_2, \dots),$$

so that, by equation (1) of § 3,

$$\left\{y_1^2\frac{d}{dy_1} - y_1\left(2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + \dots\right) - V\right\}(y_2^{-k}R) = \pm\frac{d}{dx_1}(x_2^{-k}R).$$

But having regard to the value of $U(y_2^{-k}R)$, as given in (7) of § 6, this may be written

$$\left(\frac{d}{dy_1} - U\right)(y_2^{-k}R) = \pm\frac{d}{dx_1}(x_2^{-k}R),$$

so that
$$U(y_2^{-k}R) = y_2^{-k}\frac{dR}{dy_1} \mp x_2^{-k}\frac{dR}{dx_1}.$$

This last equation shows that, if R is an orthogonal reciprocant, $\frac{dR}{dy_1}$ must be a reciprocant; and that, conversely, if R is a reciprocant such that $\frac{dR}{dy_1}$ is also a reciprocant, then R must be an orthogonal one. These are of course well-known results, due to Professor Sylvester.

9. In § 3, let F stand for $y_1^{-(2n-1)}\,Y_n$ (Y_n was defined in § 1); then $\Phi = x_n$, thus equation (1) of § 3 will give

$$\left[-y_1^2\frac{d}{dy_1} + y_1\left(2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + \dots\right) + V\right]\left[Y_n y_1^{-(2n-1)}\right] = 0,$$

or
$$\left\{-y_1^2\frac{d}{dy_1} + y_1\left(2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + \dots\right) + V\right\}Y_n = (2n-1)\,y_1 Y_n,$$

or
$$-y_1^2\frac{dY_n}{dy_1} + (n-1+2n-2)\,y_1 Y_n + VY_n = (2n-1)\,y_1 Y_n$$

[since Y_n is of degree $n-1$ and weight $2(n-1)$],

which reduces to

$$y_1^2\frac{dY_n}{dy_1} - (n-2)\,y_1 Y_n - VY_n = 0 \quad\dots\dots\dots\dots\dots\dots(10).$$

10. The equation (10) may be put into a very simple form in the following manner. But at this point it is convenient to abandon the notation used so far, and to take the usual one; I write, then, t in place of y_1, and $a, b, c, \ldots$ in place of $y_2, y_3, y_4, \ldots$. This done, (10) takes the form

$$t^2\frac{dY_n}{dt}-(n-2)\,tY_n-VY_n=0 \quad \text{..................}(11).$$

Let Y_n be written in the form

$$t^{n-2}A_0+t^{n-3}qA_1+t^{n-4}q^2A_2+\ldots+q^{n-2}A_{n-2},$$

where $A_0, A_1, \ldots$ are *pure* functions (*i.e.*, they do not involve t), and $q=1$ is a quantity put in to make the expression homogeneous. Then, since Y_n is homogeneous and of degree $n-2$, if considered as a quantic in t and q,

$$t\frac{dY_n}{dt}+q\frac{dY_n}{dq}=(n-2)\,Y_n,$$

therefore

$$t^2\frac{dY_n}{dt}-(n-2)\,tY_n+tq\frac{dY_n}{dq}=0,$$

subtracting which from (11) (in which the VY_n must be multiplied by q to make the equation homogeneous), we see that the latter takes the very simple form

$$VY_n=-t\frac{dY_n}{dq} \quad \text{..............................}(12).$$

The effect of the operator U on Y_n may also be noticed; we have

$$U.\,Y_n=(1-t^2)\frac{dY_n}{dt}+3\,(n-1)\,tY_n+VY_n=\frac{dY_n}{dt}+(2n-1)\,tY_n\ldots(13),$$

substituting for VY_n from (11).

11. It is clear that by means of the Y functions any number of reciprocants can be formed. For, if we take any homogeneous and isobaric function of $Y_m, Y_n, Y_p, \ldots$ and add to (or subtract from) it the same function of $y_m, y_n, y_p, \ldots$ multiplied by any power of y_1 or t, we have an expression which does not change in value when y and x are written one for the other; *i.e.*, a reciprocant. But there will be a change in *sign* in those expressions which are obtained by subtraction; those obtained by addition will be unaltered even in sign when x and y are interchanged. That is to say, the addition method will give reciprocants of positive character, and the subtraction method reciprocants of negative character.

The simplest set of reciprocants which can be formed in this way

are obtained by adding y_n multiplied by any power of y_1 to Y_n, and by subtracting the same expressions. If X_n denote the same function of $x_1, x_2, \ldots$ that Y_n is of $y_1, y_2, \ldots$, we have

$$\begin{aligned} \pm y_n y_1^\lambda + Y_n &= \pm \frac{X_n}{x_1^{2n-1}} \frac{1}{x^\lambda} + x_n y_1^{2n-1} \\ &= y_1^{2n+\lambda-1} \{ \pm X_n + x_n x_1^\lambda \} \\ &= \pm y_1^{2n+\lambda-1} \{ \pm x_n x_1^\lambda + X_n \}. \end{aligned}$$

If, then, $y_n t^\lambda$ be added to (subtracted from) Y_n, the result is a reciprocant of positive (negative) character, and of index $2n+\lambda-1$.

Writing down the Y's in the ordinary notation,

$$\begin{aligned} Y_1 &= 1, \\ Y_2 &= -a, \\ Y_3 &= -tb+3a^2, \\ Y_4 &= -t^2c+10tab-15a^3, \\ Y_5 &= -t^3d+t^2(15ac+10b^2)-105ta^2b+105a^4, \end{aligned}$$

it is seen at once that, *e.g.*, $-tb+Y_3$ is the Schwarzian, t^2c+Y_4 is $5a$ times the Schwarzian,— $tc+Y_4$ is $2t$ times the post-Schwarzian less the negative reciprocant $15a^3$, while $-b+tY_3$ and $-c+Y_4$ are well-known orthogonal reciprocants, &c., &c. If λ be chosen so as to be equal to $n-2$, we derive the most important species of reciprocants belonging to this class, viz., the homogeneous ones. They form the series

$$\begin{aligned} N_2 &= -2a, \\ N_3 &= -2tb+3a^2, \\ N_4 &= -2t^2c+10tab-15a^3, \text{ \&c.,} \end{aligned}$$

all of negative character; and

$$\begin{aligned} P_3 &= 3a^2, \\ P_4 &= 10tab-15a^3, \\ P_5 &= 15t^2ac+10t^2b^2-105ta^2b+105a^4, \text{ \&c.,} \end{aligned}$$

all of positive character.

N_n and P_n may with fitness be called the *fundamental* mixed homogeneous reciprocants (of negative and positive character respectively) of order n.

12. The equations giving the values of the Y's in terms of y_1, y_2, &c., may be written in the form

$$y_2 = - Y_2,$$
$$y_1 y_3 = - Y_3 + 3y_2^2,$$
$$y_1^2 y_4 = - Y_4 + \text{a function of } y_1 y_3 \text{ and } y_2,$$
$$y_1^3 y_5 = - Y_5 + \text{a function of } y_1^2 y_4,\ y_1 y_3, \text{ and } y_2,$$
$$\&\text{c.}, \qquad \&\text{c.},$$

and, generally,

$$y_1^{n-2} y_n = - Y_n + \text{a function of } y_1^{n-3} y_{n-1},\ y_1^{n-4} y_{n-2},\ \ldots, \text{ and } y_2.$$

Accordingly, by successive substitutions, $y_1^{n-2} y_n$ may be expressed as a function of Y_n, Y_{n-1}, ..., Y_2. It follows that any homogeneous isobaric function f of y_1, y_2, ... y_n can, by successive substitutions, be expressed as a function ϕ of Y_2, Y_3, ... Y_n, divided by some power of Y_1; and, since $Y_1 = 1$, such function can be made homogeneous and isobaric by suitably inserting various powers of Y_1. If

$$f(y_1, y_2, \ldots y_n) = y_1^{-\lambda} \phi(Y_1, Y_2, \ldots Y_n),$$

it is readily seen that $\lambda = w - 2i$, where i, w are the degree and weight of f, considering y_r as of weight r.

For any term $y_m^{\alpha} y_n^{\beta} y_p^{\gamma} \ldots$ in f will give rise (among others) to a term

$$y_1^{-\alpha(m-2)} Y_m^{\alpha} y_1^{-\beta(n-2)} Y_n^{\beta} y_1^{-\gamma(p-2)} Y_p^{\gamma} \ldots$$

in ϕ. But this is $Y_m^{\alpha} Y_n^{\beta} Y_p^{\gamma} \ldots$ divided by y_1 raised to the power

$$(m\alpha + n\beta + p\gamma \ldots) - 2(\alpha + \beta + \gamma + \ldots),$$

i.e., to the power $w - 2i$.

We may then write

$$y_1^{w-2i} f(y_1, y_2, \ldots y_n) = \phi(Y_1, Y_2, \ldots Y_n) \quad \ldots\ldots\ldots\ldots(14).$$

The expression on the left of (14) is such that its weight is double its degree (as is the case with the Y functions). For the weight is $w - 2i + w$, that is, $2(w - i)$; and the degree is $w - 2i + i$, that is, $w - i$. Consequently ϕ will satisfy the relations

$$(w - i)\phi = Y_1 \frac{d\phi}{dY_1} + Y_2 \frac{d\phi}{dY_2} + Y_3 \frac{d\phi}{dY_3} + \&\text{c.} \quad \ldots\ldots\ldots\ldots(15),$$

$$2(w - i)\phi = Y_1 \frac{d\phi}{dY_1} + 2Y_2 \frac{d\phi}{dY_2} + 3Y_3 \frac{d\phi}{dY_3} + \&\text{c.} \quad \ldots\ldots\ldots(16).$$

In particular, if f be a reciprocant R of degree i and weight w, then, as in § 2,

$$R(y_1, y_2, \dots y_n) = \pm y_1^{w+i} R(x_1, x_2, \dots x_n)$$
$$= \pm y_1^{w+i} R(Y_1 y_1^{-1}, Y_2 y_1^{-3}, \dots Y_n y_1^{-(2n-1)})$$
$$= \pm y_1^{w+i} y_1^{-(2w-1)} R(Y_1, Y_2, \dots Y_n),$$

since any term in R such as $y_m^{\alpha} y_n^{\beta} \dots$ gives rise to a term

$$Y_m^{\alpha} y_1^{-(2m-1)\alpha} Y_n^{\beta} y_1^{-(2n-1)\beta} \dots \quad \text{or} \quad Y_m^{\alpha} Y_n^{\beta} y_1^{-(2w-i)};$$

therefore $\quad y_1^{w-2i} R(y_1, y_2, \dots y_n) = \pm R(Y_1, Y_2, \dots Y_n)$

$$= \pm Y_1^{w-2i} R(Y_1, Y_2, \dots Y_n) \dots\dots\dots (17),$$

(since $Y_1 = 1$); *i.e.*, the reciprocant on the left-hand side of (17), when expressed in terms of the Y's, takes exactly the same form, except for a possible change of sign.

As an example, take the reciprocant $y_1 y_4 - 5y_2 y_3$, of degree 2 and weight 5. We have

$$y_1 y_4 - 5y_2 y_3 = -y_1^{5+2}(x_1 x_4 - 5x_2 x_3)$$
$$= -y_1^7 y^{-(2.5-2)}(Y_1 Y_4 - 5Y_2 Y_3)$$
$$= -y_1^{-1}(Y_1 Y_4 - 5Y_2 Y_3);$$

therefore $\quad y_1^2 y_4 - 5y_1 y_2 y_3 = -(Y_1^2 Y_4 - 5Y_1 Y_2 Y_3),$

where each expression is of degree 3 and weight 2×3 in its coefficients.

From what has been said above, it is clear that any homogeneous isobaric function of $Y_2, Y_3, \dots Y_n$ (of degree i' and weight w', taking Y_r as of weight r') can be expressed as a function of $y_1, y_2, \dots y_n$ of a similar kind. If this be done, the highest power of y_1 which will occur is the $w' - 2i'^{\text{th}}$. For the highest power of y_1 which occurs in Y_r is y_1^{r-2}; therefore the highest power of y_1 in $Y_m^{\alpha} Y_n^{\beta} \dots$ will be the $(m\alpha + n\beta + \dots) - 2(\alpha + \beta + \dots)^{\text{th}}$; that is, the $w' - 2i'^{\text{th}}$.

13. Referring back to § 10, let us write

$$Y_{n+2} = t^n A_0 + t^{n-1} q A_1 + t^{n-2} q^2 A_2 + \dots + q^n A_n,$$

A_0, A_1, &c. being homogeneous functions of $y_2, y_3, \dots y_n$, and not involving t or y_1 (A_0 is in fact $-y_{n+2}$); and $q = 1$ being inserted to

make the expression homogeneous. Then it has been proved that

$$VY_{n+2} = -t\frac{dY_{n+2}}{dq}.$$

But $$VY_{n+2} = t^n VA_0 + t^{n-1}qVA_1 + t^{n-2}q^2VA_2 + \dots + q^nVA_n,$$

and $$t\frac{dY_{n+2}}{dq} = t^nA_1 + 2t^{n-1}qA_2 + 3t^{n-2}q^2A_3 + \dots + ntq^{n-1}A_n.$$

Equating coefficients of the various powers of t, we have

$$\left.\begin{aligned} VA_0 &= -A_1 \\ VA_1 &= -2A_2 \\ VA_2 &= -3A_3, \text{ \&c.} \\ VA_{n-1} &= -nA_n \\ \text{and, finally,}\qquad VA_n &= 0 \end{aligned}\right\} \dots\dots\dots\dots(18),$$

and we may write

$$Y_{n+2} = t^nA_0 - t^{n-1}qVA_0 + \frac{1}{1.2}t^{n-2}q^2V^2A_0 - \dots + \frac{(-1)^{n-1}}{\underline{|n-1}}tq^{n-1}V^{n-1}A_0,$$

where A_0 stands for $-y_{n+2}$; or, symbolically, and replacing $n+2$ by n,

$$Y_n = -t^{n-2}e^{-V/t}.y_n \dots\dots\dots\dots(19).$$

The Y functions are therefore of such a kind that, regarded as quantics $(A_0, A_1, \dots A_n \between t, q)^n$, their coefficients satisfy relations (18) of a kind precisely analogous to those satisfied by covariants in the ordinary theory of the binary quantics—the operator V here replacing the operator $a\delta_b + 2b\delta_c + 3c\delta_d + \text{\&c.}$ of the latter theory; in fact they are *quasi-covariants*, so to speak. The term A_0 may be called the *source* of the quasi-covariant Y_{n+2}; and, just as in Salmon's *Higher Algebra*, p. 127, it is seen that the source of the product of two quasi-covariants is equal to the product of their sources.

The Y's satisfy also the equation

$$WY_n = t\frac{dY_n}{dt},$$

where $$W = b\delta_b + 2c\delta_c + 3d\delta_d + \dots.$$

This is a consequence of Y_n being of degree $n-1$ and weight $2(n-1)$; for

$$(n-1)\,Y_n = (t\delta_t + a\delta_a + b\delta_b + c\delta_c + \dots)\,Y_n,$$

$$2(n-1)\,Y_n = (t\delta_t + 2a\delta_a + 3b\delta_b + 4c\delta_c + \dots)\,Y_n,$$

therefore $$0 = (-t\delta_t + b\delta_b + 2c\delta_c + \dots)\, Y_n,$$

or $$WY_n = t\frac{dY_n}{dt} \quad\dots\dots\dots\dots\dots\dots\dots\dots(20).$$

This equation and $$VY_n = -t\frac{dY_n}{dq}$$

are the analogues of the equations

$$\Omega F = y\frac{dF}{dx}, \quad OF = x\frac{dF}{dy},$$

satisfied by the covariants of a binary quantic.

14. With the same notation as in § 13, let f be any homogeneous and isobaric function of $A_0, A_1, \dots A_n$. Then

$$Vf(A_0, A_1, \dots A_n) = \frac{df}{dA_0}VA_0 + \frac{df}{dA_1}VA_1 + \dots + \frac{df}{dA_n}VA_n$$

$$= -\left\{A_1\frac{d}{dA_0} + 2A_2\frac{d}{dA_1} + \dots + nA_n\frac{d}{dA_{n-1}}\right\}f \dots\dots(21).$$

Now let Y_{n+2} or $(A_0, A_1, \dots A_n \between t, q)^n$ be regarded as a purely algebraic form, a quantic in t, q of the n^{th} degree, of which A_0, A_1, &c. are the coefficients. Then the vanishing of the right-hand side of (21) is the condition that f should be a seminvariant of the quantic, in the sense of being unaltered if q be changed into $q+\lambda$. For the expression within the brackets is precisely the second (O) of the two well-known operators (Salmon, *Higher Algebra*, § 65) written with non-binomial coefficients. The vanishing of the left-hand side of (21) is the necessary and sufficient condition that f should be a pure reciprocant. It follows that, when f is a seminvariant of

$$(A_0, A_1, \dots \between t, q)^n$$

in the sense explained (and of course also when f is a full invariant of the quantic), then f is a pure reciprocant. And conversely, any pure reciprocant f is at least a seminvariant of the quantic in the sense explained. And evidently, if there be any number of quantics of the form $(A_0, A_1, \dots \between t, q)^n$ of various degrees (corresponding to various Y's), what has been said about invariants and q-seminvariants of one of them will hold good with regard to their joint invariants and q-seminvariants.

Any number of pure reciprocants can therefore be formed from the Y's by regarding any number of these as if they were a system of

covariants belonging to a binary quantic, and forming (in any of the ordinary ways known to the theory of binary forms) invariants and q-seminvariants of them.

For example, the discriminant of Y_4 gives $3ac-5b^2$; the resultant of Y_3 and Y_4 gives $9a^2d-45abc+40b^3$; if Y_5 be written

$$\alpha t^3+3\beta t^2q+3\gamma tq^2+\delta q^3,$$

the q-seminvariant $3\beta\gamma\delta-\alpha\delta^2-2\gamma^3$ gives

$$9a^2d-45abc+40b^3, \text{ \&c., \&c.}$$

If in any of the Y's the t and q be replaced by $\frac{d}{dq}$ and $-\frac{d}{dt}$, an operator will be formed whose effect on any of the Y's is to make it into a reciprocant; for example,

$$\left(b\frac{d}{dq}+3a^2\frac{d}{dt}\right)Y_4=-2at\,(3ac-5b^2),$$

and so on. And this last method is only a particular case of one (see Faà de Bruno, *Formes Binaires*, p. 251) by the application of which to any pair of Y's any number of reciprocants ("associated" quasi-covariants) can be generated.

15. The following gives another method whereby pure reciprocants can be formed in any number from the Y functions, and is simpler of application than that of § 14. The idea is an extension of that applied to binary quantics by Mr. Griffiths. Writing

$$Y'_n=p^{n-2}A_0+p^{n-3}qA_1+p^{n-4}q^2A_2+\dots$$

where A_0, A_1, &c. are still functions of a, b, c, &c., and do not involve t, but where p and q now stand for any quantities whatever which are functions of a, b, c, ..., let us see, following Mr. Griffiths' method, whether p and q can be chosen so as to turn Y'_n into a reciprocant. We have

$$\begin{aligned}VY'_n&=p^{n-2}VA_0+p^{n-3}qVA_1+p^{n-4}q^2VA_2+\dots\\&\quad+\frac{dY'_n}{dp}Vp+\frac{dY'_n}{dq}Vq\\&=\frac{dY'_n}{dp}Vp+\frac{dY'_n}{dq}Vq-p^{n-2}A_1-2p^{n-3}qA_2-3p^{n-4}q^2A_3-\text{\&c.}\\&=\frac{dY'_n}{dp}Vp+\frac{dY'_n}{dq}(Vq-p)\;\dots\dots\dots\dots(22).\end{aligned}$$

But if Y'_n is to be a reciprocant, VY'_n must vanish; accordingly the right-hand side of (22) must vanish. It follows that, if quantities p, q can be found to satisfy the relation

$$\frac{dY'_n}{dp}Vp+\frac{dY'_n}{dq}(Vq-p)=0,$$

then these quantities will, if substituted for t, q in the expression for Y_n, give rise to a reciprocant. We may then take p and q to satisfy

either $$Vp=0,\quad Vq=p \text{..........................} (23),$$

or $$Vp=0,\quad \frac{dY'_n}{dq}=0 \text{ ....} (24),$$

or $$Vq=p,\quad \frac{dY'_n}{dp}=0 \text{ ....} (25).$$

Of these (23) are the most useful. For, since the equations (23) do not, like (24) and (25), involve Y'_n, it is clear that they will give values of p and q which, when substituted in *any* of the Y's, will give reciprocants; and moreover, since V does not involve δ_t, these reciprocants will all be pure ones.

As a simple example of the application of (23), take $p=3a^2$, $q=b$; if then we put $3a^2$ for t, and b for q, in the expressions for Y_3, Y_4, Y_5, &c., as given in § 12, we get the series of pure reciprocants

$$0,\quad 3a^3(5b^2-3ac),\quad 3a^4(-9a^2d+45abc-40b^3),\ \&c., \&c.$$

16. Proceeding exactly as in the last paragraph, only taking the orthogonal operator U instead of the operator V,

$$UY'_n=\frac{dY'_n}{dp}Up+\frac{dY'_n}{dq}Uq+p^{n-2}UA_0+p^{n-3}qUA_1+\&c. \text{.......} (26).$$

Now $\quad A_0$ is of degree 1, and of weight n,

$\quad A_1$,, 2, ,, $n+1$,

$\quad A_2$,, 3, ,, $n+2$, &c.,

therefore $t(2t\delta_t+3a\delta_a+\dots)$ operating on A_0, A_1, A_2, &c., gives the results $\quad A_0t(n+1),\quad A_1t(n+3),\quad A_2t(n+5)$, &c.;

therefore $\quad UA_0=t(n+1)A_0+VA_0=t(n+1)A_0-A_1,$

$$UA_1=t(n+3)A_1+VA_1=t(n+3)A_1-2A_2,$$

and so on; thus the last part of the right-hand side of (26) is equal to

$$t\left[(n+1)A_0p^{n-2}+(n+3)A_1p^{n-3}q+\ldots\right],$$

$$-\left[A_1p^{n-2}+2A_2p^{n-3}+\ldots\right],$$

that is, to $t\left[(n+1)Y'_n+(2n-4)Y'_n-2p\dfrac{dY'_n}{dp}\right]-p\dfrac{dY}{dq}$,

and therefore

$$UY'_n=\frac{dY'_n}{dp}(Up-2pt)+\frac{dY'_n}{dq}(Uq-p)+3(n-1)tY'_n \quad \ldots(27).$$

Now let p, q be chosen so as to satisfy any one of the equations

$$\frac{dY'_n}{dp}=0,\quad Up=2pt,$$

at the same time that it satisfies any one of the two

$$\frac{dY'_n}{dq}=0,\quad Uq=p;$$

then will $$UY'_n=3(n-1)tY'_n,$$

i.e., Y'_n will become an orthogonal reciprocant such that the factor $y_2^{-(n-1)}$ will make it absolute. And, just as in § 15, we see that, if the two equations $$Up=2pt,\quad Uq=p \quad \ldots\ldots(28)$$

be chosen, then any values of p and q which satisfy them will make *all* the Y's into orthogonal reciprocants.

As a simple example, take $p=1+t^2$ and $q=t$; then substitute $1+t^2$ for t and t for q respectively in the expressions for Y_2, Y_3, Y_4, &c., in § 11; we obtain the series of orthogonal reciprocants

$$O_2=-a,$$

$$O_3=-(1+t^2)b+3ta^2,$$

$$O_4=-(1+t^2)^2c+10(1+t^2)tab-15t^2a^3,$$

and so on; and these can be made into absolute orthogonals by dividing them by a, a^2, a^3, &c., respectively.

17. The results of § 13 are very convenient in the treatment of mixed homogeneous reciprocants. For instance, we may make use of them to prove and further extend the following theorems due to

Mr. Rogers, viz., that either of the operators

$$V, \quad 2y_1^2\frac{d}{dy_1}-V,$$

acting on a mixed homogeneous reciprocant, generates another mixed homogeneous reciprocant. So far as I know, these theorems have not been rigorously proved before.

Let R be any mixed homogeneous reciprocant of degree i and weight w; R' the same made absolute by a proper power of y_1; so that

$$R'(y_1, y_2, \ldots) = y_1^{w-2i}R(y_1, y_2, \ldots).$$

Then
$$VR' = \Sigma\frac{dR'}{dY_n}VY_n$$

$$= \Sigma\frac{dR'}{dY_n}\left\{y_1^2\frac{dR'}{dy_1}-(n-2)\,y_1Y_n\right\}$$

$$= y_1^2\frac{dR'}{dy_1} \mp y_1\left\{Y_3\frac{dR'}{dY_3}+2Y_4\frac{dR'}{dY_4}+\ldots\right\} \quad \ldots\ldots\ldots(29),$$

where the R' within the brackets on the right is

$$Y_1^{w-2i}R'(Y_1, Y_2, \ldots),$$

and the double sign corresponds to that in (17).

But now, writing R' for ϕ in (15) and (16), and subtracting the double of (15) from (16),

$$-Y_1\frac{dR'}{dY_1}+Y_3\frac{dR'}{dY_3}+2Y_4\frac{dR'}{dY_4}+\ldots = 0;$$

substituting from this in (29), we have

$$VR' = y_1\left\{y_1\frac{dR'}{dy_1}\mp Y_1\frac{dR'}{dY_1}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(30),$$

where the $+$ or the $-$ sign is to be taken according as R' is of negative or positive character.

Again, from (30),

$$2y_1^2\frac{dR'}{dy_1}-VR' = y_1\left\{y_1\frac{dR'}{dy_1}\pm Y_1\frac{dR'}{dY_1}\right\} \ldots\ldots\ldots\ldots\ldots(31),$$

and, more generally, if k be any number whatever,

$$(y_1^2 \pm y_1^k)\frac{dR'}{dy_1}-VR' = \pm\, y_1^k\frac{dR'}{dy_1}\pm y_1Y_1\frac{dR'}{dY_1}$$

$$= y_1\left\{\pm\, y_1^{k-1}\frac{dR'}{dy_1}\pm Y_1^{k-1}\frac{dR'}{dY_1}\right\} \quad \ldots\ldots(32),$$

a result which includes both (30) and (31) as particular cases.

Now by an obvious extension of what has been said in § 11, it is clear that, since the expressions within the brackets on the right of (30), (31), and (32) are symmetrical in the y's and the Y's, they will be reciprocants; and they will of course still be reciprocants when multiplied by y_1. Therefore the expressions on the left of (30), (31), (32) must all be reciprocants; and the first two of these will evidently be homogeneous. Now,

$$V.R' = V.y_1^{w-2i}R = y_1^{w-2i}VR;$$

therefore, if R is a mixed homogeneous reciprocant, VR is also a mixed homogeneous reciprocant. To see that the operator

$$(y_1^2 \pm y_1^k)\frac{d}{dy_1} - V \quad \text{.........................(33)}$$

gives a reciprocant when it acts upon R (*any* homogeneous reciprocant), and not only when it acts on R', we notice that, by a simple application of (17),

$$y_1^2(3y_2y_4 - 5y_3^2) = Y_1^2(3Y_2Y_4 - 5Y_3^2);$$

raising which to a suitable power, and dividing (17) by the result, we find

$$\frac{R(y_1, y_2, \ldots)}{(3y_2y_4 - 5y_3^2)^{\frac{1}{2}w-i}} = \frac{R(Y_1, Y_2, \ldots)}{(3Y_2Y_4 - 5Y_3^2)^{\frac{1}{2}w-i}} \quad \text{...............(34).}$$

Now (32) is equally true if for R' we substitute the expressions in (34); but if we do so, then, since the operator (33) can have no effect on the denominators, we arrive at an equation exactly like (32), but with R in place of R'. It is therefore proved that the operator (33), acting on a mixed homogeneous reciprocant, produces another reciprocant. In the particular cases (30) and (31), where $k = 2$, this reciprocant is homogeneous.

Both Mr. Rogers and myself had already independently noticed that

$$(1 + y_1^2)\frac{d}{dy_1} - V,$$

operating on a mixed homogeneous reciprocant, produces another reciprocant; but the complete theorem (32) is new, so far as I know. Taking the signs on the right of (30) and (31) along with those of (17), and with what has been said in § 11, it is seen that the operator of (30) changes the character of the reciprocant on which it acts; while that of (31) leaves the character unaltered.

18. I add various properties of the functions Y_n, N_n, P_n :—

(1) The sum of the numerical coefficients in the expression for Y_n is

$$(-1)^{n-1}(n-1)\,!$$

This may be seen immediately by writing $y = e^x$ on the right-hand and $x = \log y$ on the left-hand side of the identity

$$x_n = Y_n y_1^{-(2n-1)}.$$

(2) Y_n, Y_{n+1} are connected by the equation

$$Y_{n+1} = y_1 \frac{dY_n}{dx} - (2n-1)\, y_2 Y_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (35).$$

This comes simply from differentiating the same identity, and substituting for x_{n+1} its equivalent $Y_{n+1} y_1^{-(2n+1)}$.

(3) Y_{n+1} may be derived from Y_n by the operator

$$\left(y_1 y_3 - \frac{2n-1}{n-1} y_2^2\right) \frac{d}{dy_2} + \left(y_1 y_4 - \frac{3n-1}{n-1} y_2 y_3\right) \frac{d}{dy_3}$$
$$+ \left(y_1 y_5 - \frac{4n-1}{n-1} y_2 y_4\right) \frac{d}{dy_4} + \&c.$$

This may be deduced from (35) by means of the equations,

$$y_1 \frac{dY_n}{dx} = y_1 \left\{ y_2 \frac{d}{dy_1} + y_3 \frac{d}{dy_2} + \&c. \right\},$$

$$(n-1)\, Y_n = y_1 \frac{d}{dy_1} + y_2 \frac{d}{dy_2} + \&c.,$$

$$2\,(n-1)\, Y_n = y_1 \frac{d}{dy_1} + 2y_2 \frac{d}{dy_2} + \&c.$$

(4) N_n, N_{n+1} are connected by the equation

$$N_{n+1} = y_1 \frac{dN_n}{dx} - (2n-1)\, y_2 N_n - (n+1)\, y_1^{n-2} y_2 y_n \quad \ldots\ldots (36).$$

For, since
$$N_n = Y_n - y_1^{n-2} y_n,$$

$$\frac{dN_n}{dx} = \frac{dY_n}{dx} - (n-2)\, y_1^{n-3} y_2 y_n - y_1^{n-2} y_{n+1},$$

and
$$N_{n+1} = Y_{n+1} - y_1^{n-1} y_{n+1};$$

substituting from which in (35), the result (36) follows.

(5) P_n, P_{n+1} are connected by the equation

$$P_{n+1} = y_1 \frac{dP_n}{dx} - (2n-1)\, y_2 P_n + (n+1)\, y_1^{n-2} y_2 y_n.$$

This is found in the same way as (36).

19. The method of § 3 can easily be extended to Mr. Elliott's ternary, &c. reciprocants; but the results are somewhat complicated. Let F be any function of $\frac{dz}{dx}$, $\frac{dz}{dy}$, $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dx\,dy}$, &c., where x, y are independent variables, z a dependent variable. The effect on F of changing x into $x-\theta z$, and y into $y-\phi z$ (where θ and ϕ are infinitesimals), can be expressed without difficulty. If

$$\theta z \frac{dz}{dx} + \phi z \frac{dz}{dy} = \omega,$$

then (see, *e.g.*, Todhunter, *History of the Calculus of Variations*)

$$\delta \frac{dz}{dx} = \frac{d^2z}{dx^2}\theta z + \frac{d^2z}{dx\,dy}\phi z + \frac{d\omega}{dx},$$

$$\delta \frac{dz}{dy} = \frac{d^2z}{dx\,dy}\theta z + \frac{d^2z}{dy^2}\phi z + \frac{d\omega}{dy},$$

and so on; writing (m, n) to denote $\frac{d^{m+n}z}{dx^m\,dy^n}$, the general formula is

$$\delta(m, n) = (m+1, n)\,\theta z + (m, n+1)\,\phi z + \frac{d^{m+n}\omega}{dx^m\,dy^n},$$

and then
$$\delta F = \Sigma \frac{dF}{d(m, n)}\,\delta(m, n).$$

Since the changes in x and y are quite arbitrary, and independent of one another, the parts of δF which involve θ and ϕ respectively can be calculated independently. We shall thus find

$$\delta F = \theta \delta_\theta F + \phi \delta_\phi F,$$

where $$\delta_\theta F = \Sigma \frac{dF}{d(m,n)} \left\{ (m+1, n)\, z + \frac{d^{m+n}}{dx^m\, dy^n} \left(z \frac{dz}{dx} \right) \right\} \quad \text{......(37)},$$

$$\delta_\phi F = \Sigma \frac{dF}{d(m,n)} \left\{ (m, n+1)\, z + \frac{d^{m+n}}{dx^m\, dy^n} \left(z \frac{dz}{dy} \right) \right\} \quad \text{......(38)}.$$

If then F become, by transforming it so as to make x the dependent, and y, z the independent variables, a function Φ of $\frac{dx}{dy}$, $\frac{dx}{dz}$, $\frac{d^2x}{dy^2}$, &c., then, exactly as in § 3, it is seen that the partial differential coefficient of Φ with respect to $\frac{dx}{dz}$ is equal to the expression on the right of (37). And, again, if F become by a similar transformation a function Ψ of $\frac{dy}{dx}$, $\frac{dy}{dz}$, $\frac{d^2y}{dx^2}$, &c., then the differential coefficient of Ψ with respect to $\frac{dy}{dz}$ will be equal to the expression on the right of (38).

If F be a reciprocant, it must then clearly satisfy *two* relations of a kind analogous to equation (1) of § 3; and these can be written down without difficulty for the case of any special class of ternary reciprocants. Similar reasoning applies to the case of n-ary reciprocants; these will satisfy $n+1$ independent relations of this kind.

20. Pure ternary reciprocants will then possess a pair of annihilators. Referring to § 3, it is seen that the process of calculating V for ordinary pure reciprocants may be arranged as follows :—

$$\begin{array}{lll|l}
\omega & = & yy_1\theta & \\
\omega' & = & (yy_2 + y_1^2)\,\theta & \\
\omega'' & = & (yy_3 + 3y_1y_2)\,\theta & \\
\omega''' & = & (yy_4 + 4y_1y_3 & + 3y_2^2)\,\theta \\
\omega'''' & = & (yy_5 + 5y_1y_4 & + 10y_2y_3)\,\theta,
\end{array}$$

and so on; and the part on the right of the vertical line gives θ times V. In precisely the same manner the pair of annihilators for pure ternary reciprocants can be calculated. We have only to write down $z\frac{dz}{dx}$, and differentiate it any number of times for x or y, cutting off after differentiation all terms involving z, $\frac{dz}{dx}$, or $\frac{dz}{dy}$. What remains will give the annihilator corresponding to the change of x into $x-\theta z$. And a similar process applied to $z\frac{dz}{dy}$ will give the second annihilator,

that corresponding to the change of y into $y-\phi z$. I have only had the courage to calculate a few terms of the x-annihilator; these I give below. The corresponding terms of the y-annihilator can be derived from them by symmetry.

$$\omega = z\frac{dz}{dx} = z\,(10),$$

$$\frac{d\omega}{dx} = z\,(20)+(10)^2,\quad \frac{d\omega}{dy} = z\,(11)+(10)(01);$$

$$\frac{d^2\omega}{dx^2} = z\,(30)+3\,(10)(20),\quad \frac{d^2\omega}{dy^2} = z\,(12)+2\,(11)(01)+(10)(02),$$

$$\frac{d^2\omega}{dx\,dy} = z\,(21)+2\,(10)(11)+(01)(20);$$

$$\frac{d^3\omega}{dx^3} = 3\,(20)^2+\ldots,$$

$$\frac{d^3\omega}{dx^2\,dy} = 3\,(11)(20)+\ldots,$$

$$\frac{d^3\omega}{dx\,dy^2} = (20)(02)+2\,(11)^2+\ldots,$$

$$\frac{d^3\omega}{dy^3} = 3\,(11)(02)+\ldots;$$

$$\frac{d^4\omega}{dx^4} = 10\,(20)(30)+\ldots,$$

$$\frac{d^4\omega}{dx^3\,dy} = 4\,(11)(30)+6\,(20)(21)+\ldots,$$

$$\frac{d^4\omega}{dx^2\,dy^2} = (02)(30)+3\,(20)(12)+6\,(11)(21)+\ldots,$$

$$\frac{d^4\omega}{dx\,dy^3} = (20)(03)+3\,(02)(21)+6\,(11)(12)+\ldots,$$

$$\frac{d^4\omega}{dy^4} = 4\,(11)(03)+6\,(02)(12)+\ldots;$$

and so on, the omitted part being in each case that involving z or (10) or (01). The annihilator will therefore be

$$3\,(20)^2\frac{d}{d\,(30)}+3\,(11)(20)\frac{d}{d\,(21)}+\left\{(20)(02)+2\,(11)^2\right\}\frac{d}{d\,(12)}$$
$$+3\,(02)(11)\frac{d}{d\,(03)}+4\,(20)(30)\frac{d}{d\,(40)}+\&c.;$$

the coefficient of $\frac{d}{d\,(mn)}$ being

$$\frac{d^{m+n}\{z\,(10)\}}{dx^m\,dy^n} - \text{terms in this which involve } z, (10), \text{ or } (01).$$

In a similar manner, by following the method of § 6, the pair of operators for "orthogonal" ternary reciprocants, analogous to the operator U of § 6, might be worked out; the one by writing $x - z\theta$ for x and $z + x\theta$ for z simultaneously, and the second by writing $y - z\phi$ for y and $z + y\phi$ for z simultaneously. But the calculation would be very laborious.

21. The method of § 11 is clearly applicable, *mutatis mutandis*, to ternary reciprocants. As an example, take one of the simplest cases, and let a_1', b_1', c_1', a_1'', b_1'', c_1'' be each expressed in terms of p, q, a_1, b_1, c_1. (For the notation I refer to Mr. Elliott's paper, *Proceedings*, Vol. XVII., p. 172.) It is found that

$$\left.\begin{aligned} -a_1' &= q^2a_1 - 2pqb_1 + p^2c_1 \\ -b_1' &= pb_1 - qa_1 \\ -c_1' &= a_1 \end{aligned}\right\} \div p^3,$$

$$\left.\begin{aligned} -a_1'' &= c_1 \\ -b_1'' &= qb_1 - pc_1 \\ -c_1'' &= q^2a_1 - 2pqb_1 + p^2c_1 \end{aligned}\right\} \div q^3.$$

Then $a_1'p^3 + a_1''q^3 - a_1$ and $c_1'p^3 + c_1''q^3 - c_1$ each give the reciprocant

$$(1+q^2)\,a_1 - 2pqb_1 + (1+p^2)\,c_1,$$

while $b_1'p^3 + b_1''q^3 - b_1$ gives the reciprocant

$$(1+p+q)\,b_1 - qa_1 - pc_1.$$

These two reciprocants correspond to those obtained by the addition method of § 11. Others can be formed, involving the imaginary cube roots of unity, corresponding to those found by the subtraction method of § 11. I have not pursued this method further; but it is evidently one which may be expected to yield good results, giving, as it does, the means of forming any number of ternary reciprocants.

Homographic and Circular Reciprocants. *By* L. J. Rogers, B.A.

[*Read March* 11*th*, 1886.]

Contents of Paper.

In the following pages the following abbreviations will always be employed:

$$t,\ a,\ b,\ c,\ \dots \text{ denote } \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \dots \text{ respectively,}$$

n denotes the characteristic of a reciprocant, and i the degree, in cases where the reciprocant is homogeneous.

As I shall have to refer to homogeneous and orthogonal reciprocants, it will not be out of place to state first the manner in which successive educts in each system are formed. In every case these unreduced educts will be employed as the protomorphs of the corresponding system. We shall begin with the simplest forms, viz., the mixed homogeneous, and pass on to pure, and then orthogonal forms.

§ 1. The formation of mixed homogeneous reciprocants has been explained by Prof. Sylvester in *Mess. of Math.*, Sept., 1885. The simplest absolute reciprocant of the system is $a/t^{\frac{3}{2}}$, and the successive educts are formed by operating with $\frac{1}{t^{\frac{1}{2}}}\frac{d}{dx}$, each operation giving a new absolute reciprocant.

Now, if we take such a function μ that

$$\left(\frac{d\mu}{dx}\right)^2 = \frac{dy}{dx},$$

this operation is $\frac{d}{d\mu}$, and, if we consider its effect on a reciprocant of characteristic n, and omit the denominator $t^{\frac{3}{2}}$, it is equivalent to $t\delta_x - \frac{1}{2}na$, raising the characteristic 3.

We write then
$$\frac{d}{d\mu} = t\frac{d}{dx} - \frac{1}{2}na,$$

and define $M_1, M_2 \dots$ by the following equations :—

$$\left.\begin{aligned} M_1 &= a & (n &= 3) \\ M_2 &= tb - \tfrac{3}{2}a^2 & (n &= 6) \\ 2M_3 &= 2t^2c - 10tab + 9a^3 & (n &= 9) \\ 4M_4 &= 4t^3d - 30t^2ac - 20t^2b^2 + 124ta^2b - 81a^4 & (n &= 12) \end{aligned}\right\} \dots\dots(1),$$

&c.,

so that
$$\frac{dM_r}{d\mu} \equiv M_{r+1}.$$

In this system every M is negative in character; and therefore every reciprocant function of the M's must be either of an odd or an even degree in every term, though not necessarily homogeneous.

Thus $2M_3 - 9M_1^3$ is a reciprocant (the post-Schwarzian), the degree being odd throughout, but $M_4 + M_2^2$ is not; n is given by the equation

$$n = 2t\delta_t + 3a\delta_a + 4b\delta_b + \dots,$$

and in an irreducible function of the M's, $n = 3w$, where w is the common weight of each term; but, if the M-function contain a factor t^ν the characteristic of the remaining factor is $3w - 2\nu$, since the characteristic of t is 2. As we only concern ourselves with this remaining factor, we shall write

$$\left.\begin{aligned} n &= 3w - 2\nu \\ i &= w - \nu \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots(2).$$

§ 2. *Pure Reciprocants.*

Pure reciprocants are formed by the successive operation of $a^{-\frac{1}{3}}\frac{d}{dx}$ on any absolute pure reciprocant.

If we take a function ρ such that

$$\left(\frac{d\rho}{dx}\right)^3 = \frac{d^2y}{dx^2},$$

this operation is $\frac{d}{d\rho}$, and is equivalent to the operation of $a\delta_x - \frac{n}{3}b$ on a pure reciprocant of character n. Hence

$$\frac{d}{d\rho} = a\delta_x - \frac{n}{3}b \qquad\qquad (1),$$

omitting denominators as in § 1.

Operating successively with $\frac{d}{d\rho}$ on the well-known pure reciprocant $3ac-5b^2$, which we shall call $3R_2$, we get the following system of pure educts,

$$\left.\begin{array}{ll} 3R_2 = 3ac-5b^2 & (n=8) \\ 9R_3 = 9a^2d-45abc+40b^3 & (n=12) \\ 9R_4 = 9a^3-9a^2(7bd+5c^2)+255ab^2c-160b^4 & (n=16) \end{array}\right\} \qquad (2),$$

&c.,

so that $$\frac{dR_r}{d\rho} \equiv R_{r+1}.$$

In this system any R with an even suffix is positive, and any with an odd suffix is negative in character. Any isobaric function of the R's is a reciprocant, and there is no restriction as in the case of mixed reciprocants.

As in the homogeneous mixed system, we have

$$n = 3a\delta_a + 4b\delta_b + 5c\delta_c + \ldots,$$

and in an irreducible R-function $n = 4w$; but, if a^ν occur as a factor,

$$\left.\begin{array}{l} n = 4w-3\nu \\ i = w-\nu \end{array}\right\} \qquad (3).$$

Any pure reciprocant is annihilated by V where

$$V = 3a^2\delta_b + 10ab\delta_c + (15ac+10b^2)\delta_d + \ldots,$$

in which $10ab = \delta_x(3a^2)+4ab$, $15ac+10b^2 = \delta_x(10ab)+5ac$;

the next coefficient $= \delta_x(15ac+10b^2)+6ad$, &c.

§ 3. *Orthogonal Reciprocants.*

In this system the generating operation is

$$(1+t^2)^{-\frac{1}{2}}\frac{d}{dx}, \quad \text{or} \quad \frac{d}{ds},$$

where $$\left(\frac{ds}{dx}\right)^2 = 1+t^2.$$

Acting on the numerator of an absolute orthogonal, $\frac{d}{ds}$ is equivalent to $(1+t^2)\,\delta_x - nta$, and raises the characteristic by 3. The simplest absolute orthogonal is $a\,(1+t^2)^{-\frac{3}{2}}$, which we shall call ϕ_1, being $\frac{d\phi}{ds}$, where ϕ and s are intrinsic coordinates. Or rather, omitting denominators, we have, for the system of orthogonal protomorphs,

$$\left.\begin{aligned} \phi_1 &= a & (n=3) \\ \phi_2 &= (1+t^2)\,b - 3a^2t & (n=6) \\ \phi_3 &= (1+t^2)^2\,c - 10abt\,(1+t^2) + 3\,(5t^2-1)\,a^3 & (n=9) \end{aligned}\right\} \quad \ldots\ldots(1),$$

&c.

Here also, as in mixed homogeneous reciprocants, every ϕ is negative in character, and the same remarks apply to ϕ-functions as to M-functions.

The characteristic of any ϕ-function, containing $(1+t^2)^\nu$ as a factor, is just the same as what we had in M-functions,

$$n = 3w - 2\nu,$$

and

$$i = w - \nu.$$

The differential equation for determining n is more complicated than in the previous cases. We have, in fact,

$$nt = (1+t^2)\,\delta_t + t\,(3a\delta_a + 4b\delta_b + 5c\delta_c + \ldots) + V \ldots\ldots\ldots\ldots(2),$$

where V is the pure reciprocant annihilator.

I have hitherto been obliged, for the sake of explaining my notation, to dwell at some length on facts already known; but the chief difference of notation lies in the fact, that I have used as protomorphs in every system the unreduced educts of the first and simplest form. Unless this principle be adhered to, the M-, R-, and ϕ-functions will not be isobaric, as is necessary that they should be. I have moreover always made the coefficient of the leading term in every protomorph in any system equal to unity.

One point of difference must be noticed as regards the weights of the letters t, a, b, c ... in the M and R protomorphs.

In making $a = M_1$, $tb - \frac{3}{2}a^2 = M_2$, &c., we assume the weights of t, a, b, ... to be 0, 1, 2, ... respectively; but, in writing $ac - \frac{5}{3}b^2 = R_2$, &c., we assume the weights of a, b, c, ... to be 0, 1, 2, ... respectively. This is done in order to make the characteristic of each protomorph in either system a constant numerical multiple of the weight.

As an example of the utility of these protomorphs, we can deduce some interesting properties of the operator V.

It is easy to show that any M-function can be expressed in terms of t, M_1, M_2, R_2, R_3, ...

For, let F be such a function, and let it contain letters as far as f. Then, since M_6 and R_5 are linear in f, we see that M_6 is a function of t, a, b, c, d, e, R_5. In the same way, M_5 and e are functions of t, a, b, c, d, R_4. In this way we may eliminate b, c, d, e, f from the M-function and replace them by functions of R_5, R_4, R_3, R_2, M_2.

The operation of V, therefore, on F is

$$Vt\,.\,\delta_t + VM_1\,\delta_{M_1} + VM_2\,\delta_{M_2} + VR_2\,.\,\delta_{R_2} + \dots,$$

every term of which vanishes except the third, and

$$VF = VM_2\,\frac{dF}{dM_2} = 3a^2t\,\frac{dF}{dM_2} \quad \dots\dots\dots\dots\dots\dots\dots(3).$$

Now $\dfrac{dF}{dM_2}$ will be a reciprocant, for, let F be expanded in powers of M_2, viz.,

$$F = A + BM_2 + CM_2^2 + \dots,$$

where A, B, C are reciprocants, then, if

	F	be of characteristic	n	and character	q,
so will	A	,,	n	,,	q,
	B	,,	$n-6$	,,	$-q$,
	C	,,	$n-12$	,,	q,
		&c.		&c.	

Moreover, $$\frac{dF}{dM_2} = B + 2CM_2 + 3DM_2^2 + \dots,$$

every term of which is obviously of characteristic $n-6$, and character $-q$.

Hence $3a^2t\,\dfrac{dF}{dM_2}$ or VF is a reciprocant of characteristic $n+2$ and character $-q$ ....(4).

The same reasoning applies to orthogonals, since all such can be expressed in terms of $1+t^2$, ϕ_1, ϕ_2, R_2, R_3, &c. V therefore acting on an orthogonal yields another of opposite character, and of characteristic greater by 2 than the first.

Similarly, the operation $2t^2\delta_t - V$ is the same as

$$2t^2\delta_t + 2t^2\frac{dM_2}{dt}\cdot\frac{d}{dM_2} - VM_2\cdot\frac{d}{dM_2},$$

which is easily reduced to

$$2t^2\delta_t + 2tM_2\frac{d}{dM_2} \quad\text{or}\quad 2t\left(t\delta_t + M_2\frac{d}{dM_2}\right).$$

If therefore V be expanded in forms of t and M_2, the operation

$$t\delta_t + M_2\frac{d}{dM_2}$$

will only alter the value of the numerical coefficients in the expansion, *i.e.*, it will give another reciprocant of the same kind.

Hence $2t^2\delta_t - V$ gives a new mixed homogeneous reciprocant, increasing the characteristic by 2, and not altering the character ...(5).

Now, $1-t^2$ is a negative reciprocant of characteristic 2, therefore

$$2t^2(1+t^2)\,\delta_t - (1+t^2)\,V$$

and

$$(1-t^2)\,V$$

both add 4 to n, and are positive operations, *i.e.*, do not alter the character.

Adding and dividing by t^2, whose n is 4, we get the positive operator

$$(1+t^2)\,\delta_t - V \quad\text{.................................(6)},$$

which does not alter the character.

Acting on $tb - \frac{3}{2}a^2$, we thus get

$$(1+t^2)\,b - 3a^2.$$

Acting on $tc - 5ab$, we get

$$(1+t^2)\,c - 10abt + 15a^3.$$

The operator (6) seems to be analogous to the reciprocantive operator δ_t for orthogonals; for, operating on the last two obtained orthogonals, we come back to the reciprocants we started with. The same is easily found to be true generally, if we start with a mixed homogeneous reciprocant which only contains t to the first power.

This analogy seems to foreshadow the results established in the next section.

§ 4. *Connection between Orthogonal and Mixed Homogeneous Reciprocants.*

Let ξ, η be determined by the equations

$$\xi = x + yi, \qquad \eta = y + xi,$$

and let τ, α, β represent $\dfrac{d\eta}{d\xi}, \dfrac{d^2\eta}{d\xi^2}, \ldots$.

We have then
$$\tau = \frac{t+i}{1+ti} = i\,\frac{1+t^2}{(1+ti)^2},$$

$$\frac{d\xi}{dx} = 1+ti = i^{\frac{1}{2}}\sqrt{\frac{1+t^2}{\tau}},$$

$$\alpha = \frac{2a}{(1+ti)^2}\cdot\frac{dx}{d\xi} = \frac{2a}{(1+ti)^3},$$

or
$$\frac{\alpha}{\tau^{\frac{3}{2}}} = \frac{2}{i^{\frac{3}{2}}}\cdot\frac{a}{(1+t^2)^{\frac{3}{2}}} = \frac{\kappa}{i^{\frac{1}{2}}}\cdot\frac{a}{(1+t^2)^{\frac{3}{2}}} \text{ say.}$$

Differentiating again, we get

$$\frac{\tau\beta - \frac{3}{2}\alpha^2}{\tau^{\frac{5}{2}}} = \frac{\kappa}{i^{\frac{1}{2}}}\cdot\frac{(1+t^2)\,b - 3a^2t}{(1+t^2)^{\frac{5}{2}}}\cdot\frac{dx}{d\xi},$$

or
$$\frac{\tau\beta - \frac{3}{2}\alpha^2}{\tau^3} = \frac{\kappa}{i}\cdot\frac{(1+t^2)\,b - 3a^2t}{(1+t^2)^3}.$$

By this process we form the successive orthogonal protomorphs on the right hand, and the mixed homogeneous in ξ and η on the left. Calling, therefore, the latter μ_1, μ_2, ..., we get

$$\left.\begin{aligned} \kappa\phi_1 &= i^{\frac{1}{2}}\mu_1 \\ \kappa\phi_2 &= i\mu_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and, generally,
$$\kappa\phi_n = i^{\frac{1}{2}n}\mu_n,$$

where
$$\kappa = -2\sqrt{-1}.$$

Any homogeneous isobaric ϕ-function only differs by a constant from the corresponding μ-function, so that if any M-function be equated to zero, and its complete primitive be $y = f(x)$, then the corresponding ϕ-function will have
$$y + xi = f(x+yi) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
for its complete primitive.

Moreover, any orthogonal can be converted into a mixed homogeneous reciprocant in ξ and η. For, since a ϕ-function must be isobaric, we may neglect the powers of $i^{\frac{1}{2}}$, as we see from § 1, and since the degrees

of the terms are either all even or all odd, only even powers of κ can occur, after neglecting some power of κ, if necessary, which runs through the whole. Hence the resulting μ-function is always real. Thus $\phi_3+18\phi_1^3$ is the well-known orthogonal

$$(1+t^2)\,c-10abt+15a^3,$$

so that the corresponding μ-function is

$$\kappa^3\mu_3+18\kappa\mu_1^3=\kappa\,(\kappa^2\mu_3+18\mu_1^3),$$

or, giving κ its value $-2\sqrt{-1}$, we get

$$2\mu_3-9\mu_1^3,$$

which is easily seen to be the post-Schwarzian in ξ and η.

§ 5. A Homographic Reciprocant is a reciprocant that remains unaltered when x, y are changed into $\dfrac{Lx+M}{x+N}$, $\dfrac{L'y+M'}{y+N'}$, respectively, where $LMNL'M'N'$ are constants.

Such reciprocants, when equated to zero, give of course complete primitives of the form $$\frac{L'y+M'}{y+N'}=f\left(\frac{Lx+M}{x+N}\right) \qquad\text{(1)}.$$

They will, moreover, always be mixed and homogeneous, since we may put $\lambda x+\mu$ for x, and $\lambda' y+\mu'$ for y, without affecting the reciprocant, but we cannot put $\lambda x+\mu y+\nu$ for x, &c., so that this class cannot contain pure reciprocants.

Let us first consider differential expressions which remain unaltered when x is changed into $\dfrac{Lx+M}{x+N}$. If any function of t, a, b, ... belong to this set, and is also a reciprocant, it will be an homographic reciprocant.

It is well known that if u, y be functions of x, that

$$(tb-\tfrac{3}{2}a^2)\left(\frac{dx}{du}\right)^6$$

$$=\left\{\frac{d^3y}{du^3}\cdot\frac{dy}{du}-\tfrac{3}{2}\left(\frac{d^2y}{du^2}\right)^2\right\}\left(\frac{dx}{du}\right)^2-\left\{\frac{d^3x}{du^3}\cdot\frac{dx}{du}-\tfrac{3}{2}\left(\frac{d^2x}{du^2}\right)^2\left(\frac{dy}{du}\right)^2\right\} \quad\ldots(2).$$

Now, if $$u=\frac{Lx+M}{x+N},$$

then $$\frac{d^3x}{du^3}\cdot\frac{dx}{du}-\tfrac{3}{2}\left(\frac{d^2x}{du^2}\right)^2=0,$$

Q 2

as is easily shown, and consequently

$$\frac{tb-\frac{3}{2}a^2}{t^4} = \text{the same expression with } u \text{ instead of } x,$$

and is therefore unaltered if x be changed homographically. We may, moreover, differentiate as often as we please for y, *i.e.*, operate with $\frac{1}{t}\frac{d}{dx}$, and we shall get a series of functions which remain unaltered by this change in x. If w be the power of t in the denominator of any such function, the operation $\frac{1}{t}\frac{d}{dx}$ is evidently the same as operating with $t\delta_x - wa$ on the numerator and adding 2 to w, and w will be given by the equation

$$w = t\delta_t + 2a\delta_a + 3b\delta_b + \dots.$$

The numerators of the successive educts are

$$\left.\begin{array}{ll} tb-\frac{3}{2}a^2 & (w=4) \\ t^2c-6tab+6a^3 & (w=6) \\ \&c. & \end{array}\right\} \quad \dots\dots\dots\dots\dots\dots\dots (3).$$

Assuming as an annihilator $A\delta_a + B\delta_b + \dots = H$ say, we see that

$$H(t\delta_x - wa) = 0,$$

omitting the expression to be operated upon. That is,

$$tH\delta_x = wA = t(H\delta_x - \delta_x H),$$

because $$\delta_x H = 0.$$

But $H\delta_x - \delta_x H$ contains no differential operators of the second order, and because $\delta_x \equiv a\delta_t + b\delta_a + \dots$ we get

$$t(H\delta_x - \delta_x H) = tA\delta_t + t(B-\delta_x A)\,\delta_a + t(C-\delta_x B)\,\delta_b + \dots$$
$$= A(t\delta_t + 2a\delta_a + 3b\delta_b + \dots),$$

therefore $$t(B-\delta_x A) = 2aA, \ \&c.$$

Testing for $tb-\frac{3}{2}a^2$, we must evidently have

$$tB = 3aA,$$

therefore $$t\delta_x A = aA.$$

Integrating, we get $$A = t,$$

and because $$B-\delta_x A = 2a,$$

$$C-\delta_x B = 3b, \ \&c.,$$

we evidently get $H = t\delta_a + 3a\delta_b + 6b\delta_c + 10c\delta_d + \dots$ ....(4),

so that the particular differential functions in question are binariants, and can be converted in ordinary invariants by changing t into a; a into $2!\,b$; b into $3!\,c$, &c.

§ 6. Besides the annihilator given in § 5, homographic reciprocants will also have another annihilator which bears a remarkable analogy to V.

It is easy to show, from § 5 (2), that the expression

$$\frac{tb-\frac{3}{2}a^2}{t^2} \text{ remains unaltered on changing } y \text{ into } \frac{L'y+M'}{y+N'},$$

and consequently we may differentiate as often as we please for x, and we get a series of expressions having the same property.

The power of t in the denominator is the same as i, the degree of the numerator, therefore

$$i = t\delta_t + a\delta_a + b\delta_b + \dots,$$

and the operator for the numerator is

$$t\delta_x - ia.$$

Proceeding, as in § 5, and assuming

$$A\delta_a + B\delta_b + \dots$$

as an annihilator, we get $t\,(B-\delta_x A) = Aa,$

$$t\,(C-\delta_x B) = Ab, \text{ \&c.}$$

Testing for $tb-\frac{3}{2}a^2$, we have, as before,

$$tB = 3aA,$$

$$t\delta_x A = 2aA,$$

whence $A = t^2,$

and $B-\delta A = at,$

$$C-\delta B = bt,$$

$$D-\delta C = ct, \text{ \&c.},$$

giving the law of forming the successive coefficients. Hence the annihilator is

$$t^2\delta_a + 3at\delta_b + (4bt+3a^2)\,\delta_c + (5ct+10ab)\,\delta_d + \dots \quad \text{.........(1).}$$

Homographic Reciprocants have therefore two annihilators, but neither are sufficient to prove the reciprocant property.

§ 7. Hitherto we have obtained only one homographic reciprocant, but if we can obtain one more we can deduce an infinite number. For we can then obtain one absolute homographic reciprocant, which by differentiation for x will give another non-absolute reciprocant, also homographic. This, combined with the first to form another absolute reciprocant, will, by again differentiating for x, produce another; and the process may be carried on indefinitely.

We can easily show that

$$4M_2M_4-5M_3^2+a^2M_2^2 \quad\text{.........................}(1)$$

is homographic.

By actual calculation we easily find

$$\begin{aligned} HM_1 &= t, \\ HM_2 &= 0, \\ HM_3 &= tM_2, \\ 4HM_4 &= 10tM_3-2atM_2. \end{aligned}$$

Hence
$$H(4M_2M_4-5M_3^2+a^2M_2^2)$$
$$= M_2(10tM_3-2atM_2)-10tM_2M_3+2atM_2^2 = 0.$$

We see, therefore, that there will be an infinite number of homographic reciprocants.

§ 8. We now come to a class of reciprocants for which the most suitable name seems to be Circular Reciprocants.

We have seen, in § 4, that from any class of M-functions can be deduced a corresponding class of ϕ-functions, the latter being real only if the former be reciprocants. Now, we have obtained such a class in Homographic Reciprocants, viz.,

$$M_2,\ 4M_2M_4-5M_3^2+a^2M_2,\ \&c.,$$

see § 7, whence we see that the class of ϕ-functions

$$\phi_2,\ 4\phi_2\phi_4-5\phi_3^2-4a^2\phi_2\ \ldots\ \&c.$$

are not altered if we change

$$x+yi \text{ into } \frac{L(x+yi)+M}{x+yi+N},$$

and
$$y+xi \text{ into } \frac{L'(y+xi)+M'}{y+xi+N'}.$$

Since we may replace i by $-i$ in either of these equations, we obtain the relations that

$$x^2+y^2 \text{ becomes } \frac{(Lx+M)^2+L^2y^2}{(x+N)^2+y^2},$$

$$x \quad ,, \quad \frac{(Lx+M)(x+N)+Ly^2}{(x+N)^2+y^2},$$

$$y \quad ,, \quad \frac{(LN-M)\,y}{(x+N)^2+y^2},$$

with a further transformation obtained by interchanging x and y, and adding dashes to L, M, N.

On account of the numerators and denominators of the fractions presenting circular forms, we shall call such ϕ-functions that are annihilated by (1) Circular Reciprocants. The simplest is ϕ_3, or $(1+t^2)\,b-3a^2t$, giving the general equation to a circle as complete primitive when equated to zero.

Thursday, May 13*th*, 1886.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Mr. F. W. Watkin was admitted into the Society.

The following communications were made :—

On Cremonian Congruences contained in Linear Complexes: Dr. Hirst, F.R.S.

Solution of the Cubic and Biquadratic Equation by means of Weierstrass's Elliptic Functions: Prof. Greenhill.

On the Complex of Lines which meet a Unicursal Quartic Curve: Prof. Cayley, F.R.S.

On Airy's Solution of the Equations of Equilibrium of an Isotropic Elastic Solid under conservative forces: W. J. Ibbetson, B.A.

Conic Note: H. M. Taylor, M.A.

On the Converse of Stereographic Projection and on Contangential and Coaxal Spherical Circles: H. M. Jeffery, F.R.S.

The following presents were received :—

"Royal Society, Proceedings," Vol. XL., No. 242.

"Educational Times," for May.

"Physical Society—Proceedings," Vol. VII., Pt. 4; April, 1886.

"Solid Geometry," by Percival Frost. 3rd ed., 8vo; London, 1886.

"Annals of Mathematics," Vol. I., No. 6, Jan. 1885; Vol. II., No. 1, Sept. 1885; Charlottesville, Va.

"Bulletin de la Société Mathématique de France," T. XIV., No. 2.

"Journal de l'École Polytechnique." 55 cahier; 1885.

"Bulletin des Sciences Mathématiques," T. X.; Mai, 1886.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. II., F. 7 and 8.

"Archiv for Mathematik og Naturvidenskab," B. 10, H. 1, 2, 3, 4.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 3me Serie, T. 1, 1884, T. 2 (1re cahier), 1885.

"Mitteilungen der Mathematischen Gesellschaft," in Hamburg, No. 6; Marz, 1886.

"Tidsskrift for Mathematik," V. Raekke; 3 Aargang, 1—6 Hefte.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. VI., No. 6.

"Observations pluviométriques et thermométriques de Juin, 1883, à Mai, 1884; rapport sur les Orages de 1883." 8vo, Bordeaux, 1884, par M. Lespiault.

"Observations pluviométriques et thermométriques de Juin, 1884, à Mai, 1885; rapport sur les Orages de 1884." 8vo, Bordeaux, 1885.

C. Neumann—"Über die Kugelfunctionen P_n und Q_n, insbesondere über die Entwicklung der Ausdrücke

$$P_n\,(zz_1+\sqrt{1-z^2}\,.\,\sqrt{1-z_1^2}\cos\phi)\ \text{und}\ Q_n\,(zz_1+\sqrt{1-z^2}\,.\,\sqrt{1-z_1^2}\cos\phi),$$

nach den cosinus der Vielfachen von ϕ," (des XIII. Bandes der Abhand der Math. Phys. Classe der Königl. Sächsischen Gesellschaft der Wissenschaften, No. V.), Leipzig, 1886.

"Theory and Practice of the Slide Rule; with a short explanation of the properties of Logarithms," by Lt.-Col. J. R. Campbell, F.G.S.; Spon, 1886 (from the Author).

"Annali di Matematica," T. XIV., F. 1.

On the Complex of Lines which meet a Unicursal Quartic Curve. *By* Prof. CAYLEY.

[*Read May 13th,* 1886.]

The curve is taken to be that determined by the equations

$$x:y:z:w=1:\theta:\theta^3:\theta^4,$$

viz., it is the common intersection of the quadric surface $\Theta=0$, and the cubic surfaces $P=0$, $Q=0$, $R=0$, where

$$\Theta=xw-yz,$$

$$P=x^2z-y^3,$$

$$Q=xz^2-y^2w,$$

$$R=z^3-yw^2.$$

Writing (a, b, c, f, g, h) as the six coordinates of a line, viz.,

$$(a, b, c, f, g, h) = (\beta z-\gamma y,\ \gamma x-\alpha z,\ \alpha y-\beta x,\ \alpha w-\delta x,\ \beta w-\delta y,\ \gamma w-\delta z),$$

if $(\alpha, \beta, \gamma, \delta)$, (x, y, z, w) are the coordinates of any two points on the line; then, if the line meet the curve, we have

$$\begin{array}{rl} . \quad h\theta-g\theta^3+a\theta^4 &= 0, \\ -h \quad . \quad +f\theta^3+b\theta^4 &= 0, \\ g-f\theta \quad . \quad +c\theta^4 &= 0, \\ -a-b\theta-c\theta^3 \quad . \quad &= 0, \end{array}$$

from which four equations (equivalent, in virtue of the identity $af+bg+ch=0$, to two independent equations), eliminating θ, we have the equation of the complex; the form may, of course, be modified at pleasure by means of the identity just referred to, but one form is

$$\Omega, = a^4-b^3h+bf^2g+cg^3-acfh+2c^2h^2-4a^2ch+af^3-a^3f = 0,$$

as may be verified by substituting therein the values $a=-b\theta-c\theta^3$, $g=f\theta-c\theta^4$, $h=f\theta^3+b\theta^4$. The last-mentioned equation is thus the equation of the complex in question, in terms of the six coordinates (a, b, c, f, g, h).

If for the six coordinates we substitute their values, $\beta z-\gamma y$, &c., we obtain $\Omega, = (x, y, z, w)^4(\alpha, \beta, \gamma, \delta)^4 = 0$, which, regarded as an equation in (x, y, z, w), is the equation of the cone, vertex $(\alpha, \beta, \gamma, \delta)$, which passes through the quartic curve; this equation should evidently be satisfied if only Θ, P, Q, R are each $=0$, viz., Ω must be a linear function of (Θ, P, Q, R); and by symmetry it must be also a linear function of $(\Theta_0, P_0, Q_0, R_0)$, where

$$\begin{aligned} \Theta_0 &= \alpha\delta-\beta\gamma, \\ P_0 &= \alpha^2\gamma-\beta^3, \\ Q_0 &= \alpha\gamma^2-\beta^2\delta, \\ R_0 &= \gamma^3-\beta\delta^2, \end{aligned}$$

viz., the form is $\Omega, = (\Theta, P, Q, R)(\Theta_0, P_0, Q_0, R_0)$, an expression with coefficients which are of the first or second degree in (x, y, z, w) and also of the first or second degree in $(\alpha, \beta, \gamma, \delta)$.

To work this out, I first arrange in powers and products of (α, δ), (β, γ), expressing the quartic functions of (x, y, z, w) in terms of (Θ, P, Q, R), as follows:

$\Omega =$

	a^4	$-b^3h$	$+bf^2g$	$+cg^3$	$-acfh$	$+2c^2h^2$	$-4a^2ch$
α^4							
$\alpha^3\delta$		$-z^4$	$+yzw^2$				
$\alpha^2\delta^2$			$-2xyzw$			$+2y^2z^2$	
$\alpha\delta^3$			$+x^2yz$	$-y^4$			
δ^4							
$\alpha^3\beta$			$-zw^3$				
$\alpha^2\beta\delta$			$+2xzw^2$		$+yz^2w$		
$\alpha\beta\delta^2$			$-x^2zw$	$+3y^3w$	$-xyz^2$	$-4xyz^2$	
$\beta\delta^3$				$+xy^3$			
$\alpha^3\gamma$		$+z^3w$					
$\alpha^2\gamma\delta$		$+3xz^3$	$-xyw^2$		$-y^2zw$	$-4y^2zw$	
$\alpha\gamma\delta^2$			$+2x^2yw$		$+xyz^2$		
$\gamma\delta^3$				$-x^3y$			
$\alpha^2\beta^2$							
$\alpha^2\beta\gamma$			$+xw^3$		$-yzw^2$		
$\alpha^2\gamma^2$		$-3xz^2w$			$+y^2w^2$	$+2y^2w^2$	
$\alpha\beta^2\delta$				$-3y^2w^2$	$-xz^2w$		$+4yz^3$
$\alpha\beta\gamma\delta$			$-2x^2w^2$		$+2xyzw$	$+8xyzw$	$-8y^2z^2$
$\alpha\gamma^2\delta$		$-3x^2z^2$			$-xy^2w$		$+4y^3z$
$\beta^2\delta^2$				$-3xy^2w$	$+x^2z^2$	$+2x^2z^2$	
$\beta\gamma\delta^2$			$+x^3w$		$-x^2yz$		
$\gamma^2\delta^2$							
$\alpha\beta^3$				$+yw^3$			
$\alpha\beta^2\gamma$					$+xzw^2$		$-4yz^2w$
$\alpha\beta\gamma^2$					$-xyw^2$	$-4xyw^2$	$+8y^2zw$
$\alpha\gamma^3$		$+3x^2zw$					$-4y^3w$
$\beta^3\delta$				$+3xyw^2$			$-4xz^3$
$\beta^2\gamma\delta$					$-x^2zw$	$-4x^2zw$	$+8xyz^2$
$\beta\gamma^2\delta$					$+x^2yw$		$-4xy^2z$
$\gamma^3\delta$		$+x^3z$					
β^4	$+z^4$			$-xw^3$			
$\beta^3\gamma$	$-4yz^3$						$+4xz^2w$
$\beta^2\gamma^2$	$+6y^2z^2$					$+2x^2w^2$	$-8xyzw$
$\beta\gamma^3$	$-4y^3z$						$+4xy^2w$
γ^4	$+y^4$	$-x^3w$					

$+af^3$	$-a^3f$		
		0	
		$-z^4+yzw^2$	$-zR$
		$-2xyzw+2y^2z^2$	$-2yz\,\Theta$
		$+x^2yz-y^4$	$+yP$
		0	
$+zw^3$		0	
$-3xzw^2$		$-xzw^2+yz^2w$	$-zw\,\Theta$
$+3x^2zw$		$+2x^2zw+3y^3w-5xyz^2$	$+2xz\,\Theta-3yQ$
$-x^3z$		$+xy^3-x^3z$	$-xP$
$-yw^3$		$+z^3w-yw^3$	$+wR$
$+3xyw^2$		$+3xz^3+2xyw^2-5y^2zw$	$+2yw\,\Theta+3zQ$
$-3x^2yw$		$-x^2yw+xyz^2$	$-xy\,\Theta$
$+x^3y$		0	
		0	
		$+xw^3-yzw^2$	$+w^2\Theta$
		$-3xz^2w+3y^2w^2$	$-3wQ$
		$-3y^2w^2-xz^2w+4yz^3$	$-4z^2\Theta+3wQ$
		$-2x^2w^2+10xyzw-8y^2z^2$	$+(-2xw+8yz)\,\Theta$
		$-3x^2z^2-xy^2w+4y^3z$	$-4y^2\Theta-3xQ$
		$-3xy^2w+3x^2z^2$	$+3xQ$
		$+x^3w-x^2yz$	$+x^2\Theta$
		0	
	$-z^3w$	$+yw^3-z^3w$	$-wR$
	$+3yz^2w$	$+xzw^2-yz^2w$	$+zw\,\Theta$
	$-3y^2zw$	$-5xyw^2+5y^2zw$	$-5yw\,\Theta$
	$+y^3w$	$+3x^2zw-3y^3w$	$+3wP$
	$+xz^3$	$+3xyw^2-3xz^3$	$-3xR$
	$-3xyz^2$	$-5x^2zw+5xyz^2$	$-5xz\,\Theta$
	$+3xy^2z$	$+x^2yw-xy^2z$	$+xy\,\Theta$
	$-xy^3$	$+x^3z-xy^3$	$+xP$
		$+z^4-xw^3$	$+zR-w^2\Theta$
		$-4yz^3+4xz^2w$	$+4z^2\Theta$
		$+2x^2w^2-8xyzw+6y^2z^2$	$+(2xw-6yz)\,\Theta$
		$-4y^3z+4xy^2w$	$+4y^2\Theta$
		$+y^4-x^3w$	$-yP-x^2\Theta$

Collecting the terms multiplied by P, Q, R, Θ, respectively, we have

$$\begin{aligned}
\Omega = \ & P\left\{y\alpha\delta^3 - x\beta\delta^3 + 3w\alpha\gamma^3 + x\gamma^3\delta - y\gamma^4\right\} \\
& + Q\left\{-3y\alpha\beta\delta^2 + 3z\alpha^2\gamma\delta - 3w\alpha^2\gamma^2 + 3w\alpha\beta^2\delta - 3x\alpha\gamma^2\delta + 3x\beta^2{}^2\right\} \\
& + R\left\{-z\alpha^3\delta + w\alpha^3\gamma - w\alpha\beta^3 - 3x\beta^3\delta + z\beta^4\right\} \\
& + \Theta\left\{-2yz\alpha^2\delta^2 - zw\alpha^2\beta\delta + 2xz\alpha\beta\delta^2 + 2yw\alpha^2\gamma\delta - xy\alpha\gamma\delta^2\right. \\
& \qquad + w^2\alpha^2\beta\gamma - 4z^2\alpha\beta^2\delta + (-2xw + 8yz)\,\alpha\beta\gamma\delta - 4y^2\alpha\gamma^2\delta + x^2\beta\gamma\delta^2 \\
& \qquad + zw\alpha\beta^2\gamma - 5yw\alpha\beta\gamma\delta - 5xz\beta^2\gamma\delta + xy\beta\gamma^2\delta \\
& \qquad \left. - w^2\beta^4 + 4z^2\beta^3\gamma + (2xw - 6yz)\,\beta^2\gamma^2 + 4y^2\beta\gamma^3 - x^2\gamma^4\right\},
\end{aligned}$$

which may be written as follows :—

$$\begin{aligned}
\Omega = \ & P\left\{y\,(\alpha\delta^3 - \gamma^4) + x\,(\gamma^3\delta - \beta\delta^3)\right\} && + P\,(3w\alpha\gamma^3) \\
& + Q\left\{3x\,(\beta^2\delta^2 - \alpha\gamma^2\delta) + 3w\,(\alpha\beta^2\delta - \alpha^2\gamma^2)\right\} && + Q\,(3z\,\alpha^2\gamma\delta - 3y\,\alpha\beta\delta^2) \\
& + R\left\{-z\,(\alpha^3\delta - \beta^4) + w\,(\alpha^3\gamma - \alpha\beta^3)\right\} && + R\,(-3x\beta^3\delta) \\
& + \Theta\left\{zw\,(-\alpha^2\beta\delta + \alpha\beta^2\gamma)\right. && \\
& \qquad + xz\,2\,(\alpha\beta\delta^2 - \beta^2\gamma\delta) && + \Theta\,(-3xz\beta^2\gamma\delta) \\
& \qquad + yw\,2\,(\alpha^2\gamma\delta - \alpha\beta\gamma^2) && + \Theta\,(-3yw\,\alpha\beta\gamma^2) \\
& \qquad + xy\,(-\alpha\gamma\delta^2 + \beta\gamma^2\delta) && \\
& \qquad + xw\,2\,(-\alpha\beta\gamma\delta + \beta^2\gamma^2) && \\
& \qquad + yz\,(-2\alpha^2\delta^2 + 8\alpha\beta\gamma\delta - 6\beta^2\gamma^2) && \\
& \qquad + x^2\,(\beta\gamma\delta^2 - \gamma^4) && \\
& \qquad + y^2\,4\,(-\alpha\gamma^2\delta + \beta\gamma^3) && \\
& \qquad + z^2\,4\,(-\alpha\beta^2\delta + \beta^3\gamma) && \\
& \qquad \left. + w^2\,(\alpha^2\beta\gamma - \beta^4)\right\}, &&
\end{aligned}$$

in which all the terms contained in the $\{\ \}$ admit of expression in terms of P_0, Q_0, R_0, Θ_0; the remaining six terms not included within $\{\ \}$ may be written

$$3wP\,\alpha\,(\gamma^3 - \beta\delta^2) + 3\,(wP - yQ)\,\alpha\beta\delta^2 \quad - 3\Theta\,xz\,\beta^2\gamma\delta,$$

$$-3xR\,\delta\,(\beta^3 - \alpha^2\gamma) + 3\,(-xR + zQ)\,\alpha^2\gamma\delta - 3\Theta\,yw\,\alpha\beta\gamma^2;$$

which, observing that $wP - yQ = xz\Theta$, and $-xR + zQ = yw\Theta$, are

$$-3wP\,\alpha\,(\gamma^3 - \beta\delta^2) + 3xz\,\Theta\,(\alpha\beta\delta^2 - \beta^2\gamma\delta),$$

$$-3xR\,\delta\,(\beta^3 - \alpha^2\gamma) + 3yw\,\Theta\,(\alpha^2\gamma\delta - \alpha\beta\gamma^2).$$

The expression thus becomes

$$
\begin{aligned}
\Omega = \quad P\,. \quad & x\,(\gamma^3\delta-\beta\delta^3) && = x\,\delta R_0 \\
& +\ y\,(\alpha\delta^3-\gamma^4) && = y\,(-\gamma R_0+\delta^2\Theta) \\
& +3w\,\alpha\,(\gamma^3-\beta\delta^2) && = 3w\,\alpha R_0 \\
+Q\,. & -3x\,(\beta^2\delta^2-\alpha\gamma^2\delta) && = -3x\,\delta Q_0 \\
& +3w\,(\alpha\beta^2\delta-\alpha^2\gamma^2) && = -3w\,\alpha Q_0 \\
+R\,. & -3x\,\delta\,(\beta^3-\alpha^2\gamma) && = 3x\,\delta P_0 \\
& -\ z\,(\alpha^3\delta-\beta^4) && = z\,(-\beta P_0-\alpha^2\Theta_0) \\
& +\ w\,(\alpha^3\gamma-\alpha\beta^3) && = w\,\alpha P_0 \\
+\Theta\,. \quad & zw\,(-\alpha^2\beta\delta+\alpha\beta^2\gamma) && = -\ zw\,\alpha\beta\Theta_0 \\
& +5xz\,(\alpha\beta\delta^2-\beta^2\gamma\delta) && = 5xz\,\beta\delta\Theta_0 \\
& +5yw\,(\alpha^2\gamma\delta-\alpha\beta\gamma^2) && = 5yw\,\alpha\gamma\Theta_0 \\
& +\ xy\,(-\alpha\gamma\delta^2+\beta\gamma^2\delta) && = -\ xy\,\gamma\delta\,\Theta_0 \\
& +2xw\,(-\alpha\beta\gamma\delta+\beta^2\gamma^2) && = -2xw\,\beta\gamma\Theta_0 \\
& +\ yz\,(-2\alpha^2\delta^2+8\alpha\beta\gamma\delta-6\beta^2\gamma^2) && = -2yz\,(\alpha\delta-3\beta\gamma)\,\Theta_0 \\
& +\ x^2\,(\beta\gamma\delta^2-\gamma^4) && = -\ x^2\gamma R_0 \\
& +4y^2\,(-\alpha\gamma^2\delta+\beta\gamma^3) && = -4y^2\,\gamma^2\Theta_0 \\
& +4z^2\,(-\alpha\beta^2\delta+\beta^3\gamma) && = -4z^2\beta^2\Theta_0 \\
& +\ w^2\,(\alpha^2\beta\gamma-\beta^4) && = w^2\beta P_0
\end{aligned}
$$

and we thus finally obtain

$$
\begin{aligned}
\Omega = \quad & PR_0\,(3\alpha w-\gamma y+\delta x) \\
& +RP_0\,(3\delta x-\beta z+\alpha w) \\
& +P\Theta_0\,.\,\delta^2 y \\
& +R\Theta_0\,.\,-\alpha^2 z \\
& +P_0\Theta\,.\,\beta w^2 \\
& +R_0\Theta\,.\,-\gamma x^2 \\
& -QQ_0\,.\,-3\,(\alpha w+\delta x) \\
& +\Theta\Theta_0\,\{-\alpha\beta zw-\gamma\delta xy+5\beta\delta xz+5\alpha\gamma yw-2\beta\gamma xw-2\alpha\delta yz \\
& \qquad -4\gamma^2y^2+6\beta\gamma\,yz-4\beta^2z^2\},
\end{aligned}
$$

viz., $\Omega = 0$ is the equation of the cone, vertex $(\alpha, \beta, \gamma, \delta)$, which passes through the quartic curve $x : y : z : w = 1 : \theta : \theta^3 : \theta^4$. As regards the symmetry of this expression, it is to be remarked that, changing (x, y, z, w) and $(\alpha, \beta, \gamma, \delta)$ into (w, z, y, x) and $(\delta, \gamma, \beta, \alpha)$ respectively, we change (Θ, P, Q, R) and $(\Theta_0, P_0, Q_0, R_0)$ into $(\Theta, -R, -Q, -P)$ and $(\Theta_0, -R_0, -Q_0, -P_0)$, respectively, and so leave Ω unaltered. Again, interchanging (x, y, z, w) and $(\alpha, \beta, \gamma, \delta)$, we interchange (Θ, P, Q, R) and $(\Theta_0, P_0, Q_0, R_0)$, and so leave Ω unaltered.

Thursday, June 10th, 1886.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

SPECIAL MEETING.

The President commenced the proceedings by stating the reasons which had led the Council to summon the meeting,* and enforced the desirability of accepting the following resolution,—"That the Council be empowered to take the necessary steps to obtain a Charter of Incorporation for the Society."

The resolution, which was moved by S. Roberts, F.R.S., in a speech which gave a full and clear account of the advantages to be derived from the possession of a Charter, was seconded, on the same lines, by A. B. Kempe, F.R.S. (the Treasurer), and was carried unanimously.

The question, raised by Professor Genese, of omitting "London" from the Society's title, though it was not within the scope of the meeting, was decided, on a show of hands, by an almost unanimous vote in favour of no change being made.

The meeting then became an

ORDINARY MEETING.

The Minutes of the last meeting were read and confirmed.

Messrs. A. R. Forsyth, F.R.S., R. Lachlan, and the Rev. J. J. Milne, were admitted into the Society.

The following communications were made :—

Reciprocation in Statics : Professor Genese.

Formula for the interchange of the Independent and Dependent Variables, with some applications to Reciprocants : C. Leudesdorf, M.A.

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* Twenty-eight members were present.

Second paper on Reciprocants: L. J. Rogers, B.A.

On the Theory of Screws in Elliptic Space (Third Note): A. Buchheim, M.A.

On the motion of a Liquid Ellipsoid under the influence of its own Attraction: A. B. Basset, M.A.

Some applications of Weierstrass's Elliptic Functions: Professor Greenhill.

Electrical Oscillations on Cylindrical Conductors: Professor J. J. Thomson, F.R.S.

The following presents were received:—

"Educational Times," for June.

"Scientific Transactions of the Royal Dublin Society," Vol. III. (Ser. ii.), Parts 7 to 10.

"Scientific Proceedings of the Royal Dublin Society," Vol. IV. (N. S.), Parts 7—9; Vol. V., Parts 1, 2.

"Johns Hopkins University Circulars," Vol. V., No. 47.

"Annular Eclipse of the Sun, March 15—16, 1885." By Allan D. Brown & Albert G. Winterhalter. 4to; Washington, 1885.

"Acta Mathematica," T. 4.

"Beiblätter zu den Annalen der Physik und Chemie," B. X., St. 4.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. II., F. 9, 10, 11.

"Journal für die reine und angewandte Mathematik," Bd. 99, H. 4; Bd. 100, H. 1.

"Archives Néerlandaises des Sciences Exactes et Naturelles," T. XX., L. 4.

"Archiv for Mathematik og Naturvidenskab," B. 8, H. 3, 4; B. 9, H. 1, 2, 3, 4; B. 10, H. 1, 2, 3, 4.

"Classification der Flächen nach der Transformations gruppe ihrer Geodätischen Curven," von Sophus Lie. 4to; Kristiania, 1879.

"Niew Archief voor Wiskunde," D. XII., St. 2.

"Liste Alphabétique de la Correspondance de Christiaan Huygens qui sera publiée par la Société Hollandaise des Sciences à Harlem." 4to; Harlem.

"Register naar eene Wetenschappelijke Verdeeling op de Werken van het Wiskundig Genootschap." "Een onvermoeide Arbeid Komt alles te boven," Gedurende het tijdsverloop van 1818—1882. 8vo; Amsterdam, 1885.

"Bulletin des Sciences Mathématiques," T. X.; Juin 1886.

"Bollettino delle Pubblicazioni Italiane," ricevute per Diritto di Stampa; Firenze, 1886, Nos. 1—10.

"Lösung des Charakteristiken—Problems für lineare Räume beliebiger Dimension," von Hermann Schubert. 8vo pamphlet.

"Su le superficie di 4° ordine con conica doppia." "Memoria di H. G. Zeuthen, pubblicata per la festa commemorativa del IV. Centenario dell' Universita di Copenhagen." Giugno; 1879. Versione dal Danese, riveduta dall' autore di Gino Loria. 4to.

"Su alcune proprietà metriche della cubica gobba osculatrice al piano all'infinito, nota di Gino Loria." 4to.

On the Theory of Screws in Elliptic Space (Third Note).

By A. Buchheim, M.A.

[*Read June* 10*th*, 1886.]

This Note is a continuation of two previous Notes of mine on the same subject which have appeared in these *Proceedings* (Vol. xiv., p. 83, and Vol. xvi., p. 15). In the last two sections of the second Note, I gave some formulæ relating to infinitesimal motions, applicable to the three kinds of uniform space of three dimensions. In the present Note I consider finite motions.

Starting with Prof. Cayley's expression for an orthogonal matrix in terms of a skew matrix, I show how this is connected with the screw defining the motion. Then, transforming the matrix to its canonical form, I obtain formulæ relating to the distances and angles, through which points, lines, and planes are moved by a given screw. In this part of the paper I make use of Grassmann's methods, and of the theory of matrices, as presented in my paper on the subject in these *Proceedings* (Vol. xvi., p. 63). In the remaining part of the paper I make use of biquaternions, referring to my paper in the *American Journal of Mathematics* (Vol. vii., Pt. 4), and obtain the following theorem:—Any screw motion is represented by a biquaternion, in such wise that, if Q is a biquaternion, and ρ any bivector, $Q\rho Q^{-1}$ is the bivector into which ρ is transformed by the motion defined by Q; and, if Q is brought to the form

$$1+\alpha+\omega\,(S\alpha\alpha'+\alpha'),$$

the motion defined by Q is the motion defined by the screw $\alpha+\omega\alpha'$.

I.

In any kind of space a *motion* is a linear transformation which leaves the absolute unaltered. If we refer the absolute to a self-conjugate tetrahedron, and reduce its equation to the form

$$x^2+y^2+z^2+w^2=0,$$

we see that a motion is a linear transformation by which $x^2+y^2+z^2+w^2$ is unaltered,* that is to say, it is an orthogonal transformation, and

* To a scalar factor *près*; but we can always suppose this factor to be unity.

the condition that a matrix ϕ may be orthogonal is $\phi\phi'=1$, if ϕ' is the conjugate of ϕ.

Now, let ϕ be any orthogonal matrix, and let

$$\psi=\frac{1-\phi}{1+\phi}.$$

Then

$$\begin{aligned}\psi'&=\frac{1-\phi'}{1+\phi'}\\ &=\frac{1-\phi^{-1}}{1+\phi^{-1}}\\ &=-\frac{1-\phi}{1+\phi}\\ &=-\psi.\end{aligned}$$

That is, ψ is a skew symmetric matrix, and therefore any orthogonal matrix may be written in the form

$$\phi=\frac{1-\psi}{1+\psi},$$

where ψ is a skew symmetric matrix. For actual calculation, it is more convenient to write
$$\phi=\frac{2}{1+\psi}-1.$$

Since ϕ is a function of ψ, we see that the latent points of ϕ are the same as those of ψ, and that, if λ is a latent root of ψ, the corresponding latent root of ϕ will be
$$\frac{1-\lambda}{1+\lambda}$$

I proceed to consider the latent points and roots of a skew symmetric matrix, ψ. Let e_1, e_2 ... be the latent points, λ_1, λ_2 ... the latent roots. I shall assume that ψ is not a derogatory matrix, so that λ_1, &c. are all unequal. We have
$$Se_i\,\psi e_k=\lambda_k\,Se_i\,e_k.$$
But
$$\begin{aligned}Se_i\,\psi e_k&=Se_k\,\psi' e_i\\ &=-Se_k\,\psi e_i,\end{aligned}$$
because ψ is a skew symmetric matrix, $=-\lambda_i\,Se_i\,e_k$.

Therefore, either $\lambda_k+\lambda_i=0$ or $Se_i\,e_k=0$. Moreover, by taking $i=k$, we see that either $\lambda_i=0$ or $r^2e_i=0$. If, then, we attend only to those latent points for which the corresponding latent root does not vanish, we see that all these points are on the absolute, that they

group themselves in pairs for which the sum of the latent roots is zero, and that any two points not forming a pair are on a generator of the absolute.*

I shall now confine myself to matrices of the fourth order. Let

$$\psi = \begin{pmatrix} 0 & h & -g & a \\ -h & 0 & f & b \\ g & -f & 0 & c \\ -a & -b & -c & 0 \end{pmatrix}.$$

Then we find that the latent roots are given by

$$\lambda^4 - \lambda^2 (a^2+b^2+c^2+f^2+g^2+h^2) + (af+bg+ch)^2 = 0.$$

Now, write α^2 for $a^2+b^2+c^2+f^2+g^2+h^2$, and let

$$\sin\phi = \frac{2(af+bg+ch)}{\alpha^2};$$

then we have
$$\lambda^4 - \lambda^2\alpha^2 + \frac{\alpha^2\sin^2\phi}{4} = 0,$$

and, writing λ^2, μ^2 for the roots of this equation, we get

$$\lambda^2 = -\alpha^2\sin^2\frac{\phi}{2},$$

$$\mu^2 = -\alpha^2\cos^2\frac{\phi}{2}.$$

We see that the latent roots of ψ are

$$\pm\alpha i\sin\frac{\phi}{2}, \quad \pm\alpha i\cos\frac{\phi}{2},$$

and that, therefore, the latent roots of

$$\frac{1-\psi}{1+\psi}$$

are
$$\frac{1+\alpha i\sin\frac{\phi}{2}}{1-\alpha i\sin\frac{\phi}{2}}, \quad \frac{1+\alpha i\cos\frac{\phi}{2}}{1-\alpha i\cos\frac{\phi}{2}}, \quad \frac{1-\alpha i\sin\frac{\phi}{2}}{1+\alpha i\sin\frac{\phi}{2}}, \quad \frac{1-\alpha i\cos\frac{\phi}{2}}{1+\alpha i\cos\frac{\phi}{2}}.$$

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* For if α, β are any two points, $T^2(\lambda\alpha+\mu\beta) = \lambda^2T^2\alpha + 2\lambda\mu S\alpha\beta + \mu^2T^2\beta$, and therefore vanishes for all values of λ, μ if $T^2\alpha = S\alpha\beta = T^2\beta = 0$; and therefore, if these three conditions are fulfilled, the line $[\alpha\beta]$ is a generator of the absolute.

It will be worth while to give the actual value of

$$\Phi = \frac{1-\psi}{1+\psi}.$$

If $\beta = af+bg+ch$, $\Delta = 1+\alpha^2+\beta^2$, we find (*Crelle*, t. 32, 1846)

$$\Delta\Phi = \begin{pmatrix} 1+f^2-g^2-h^2+b^2+c^2-a^2-\beta^2, & 2(-h-ab+fg-c\beta), & 2(g-ca+hf+b\beta), & 2(-a+bh-cg-f\beta) \\ 2(h-ab+fg+c\beta), & 1+g^2-h^2-f^2+c^2+a^2-b^2-\beta^2, & 2(-f-bc+gh-a\beta), & 2(-b+cf-ah-g\beta) \\ 2(-g-ca+hf-b\beta), & 2(f-bc+gh+a\beta), & 1+h^2-f^2-g^2-c^2+a^2+b^2-\beta^2, & 2(-c+ag-bf-h\beta) \\ 2(a+bh-cg+f\beta), & 2(b+cf-ah+g\beta), & 2(c+ag-bf+h\beta), & 1-a^2-b^2-c^2+f^2+g^2+h^2-\beta^2 \end{pmatrix}$$

We have also
$$\Delta = 1+\alpha^2+\frac{\alpha^4\sin^2\phi}{4}$$
$$= \left(1+\alpha^2\cos^2\frac{\phi}{2}\right)\left(1+\alpha^2\sin^2\frac{\phi}{2}\right).$$

II.

Let $a = ae_2e_3+be_3e_1+ce_1e_2+fe_1e_4+ge_2e_4+he_3e_4$ be any screw, $x = xe_1+ye_2+ze_3+we_4$ any point.* Then, if $xa = l$, we have

$$(lmnp) = (\psi \between xyzw),\dagger$$

where ψ is the same matrix as in (1). We see, therefore, that to a given screw a appertains a certain skew symmetric matrix, and it follows from what was proved in (1) that the latent roots of the matrix are

$$\pm iTa\sin\frac{\phi}{2},\quad \pm iTa\cos\frac{\phi}{2},$$

where Ta, ϕ are the tensor and the pitch of a respectively.

It appears, therefore, that the connexion between a motion and a screw is as follows: the motion is defined by an orthogonal matrix,

---

* The general formula for any kind of space can be got from this by writing ea, eb, ec, for a, b, c in the value of Δ, and in the three first columns of the matrix.

† It will be convenient to denote screws, points, and planes by their first coordinates: thus, "the plane l" means the plane $(lmnp)$.

R 2

to this corresponds a skew symmetric matrix, and this skew matrix appertains to a certain screw. And conversely, given a screw, we can find the skew symmetric matrix appertaining to it, and then an orthogonal matrix corresponding to this skew symmetric matrix, and this orthogonal matrix defines a motion given by the screw.

We have now to reduce these matrices to their canonical forms, and to see what our metric functions become.

III.

The latent roots of the orthogonal matrix Φ have been given in (1), and we see that they are of the form α, β, α^{-1}, β^{-1}. Call the latent points $e_1e_2e_3e_4$, as before; then e_1e_2, e_1e_3, e_2e_3, e_3e_4 are generators of the absolute, and we see that, since the edges of the tetrahedron of reference are generators, the equation of the absolute must be of the form $xz-\lambda yw=0$. It will be necessary to work out the three transformed equations of the absolute.

If e_1, e_2, e_3, e_4 are four points on the absolute, such that e_1e_2, e_1e_4, e_2e_3, e_3e_4 are generators, it can be verified without difficulty that the four points

$$\eta_1=e_1+e_3,\quad \eta_2=e_2+e_4,\quad \eta_3=e_1-e_3,\quad \eta_4=e_1-e_4$$

are the vertices of a self-conjugate tetrahedron.

Now, for *any* self-conjugate tetrahedron, the fundamental metric functions are $x^2+y^2+z^2+w^2$, $l^2+m^2+n^2+p^2$, $a^2+b^2+c^2+f^2+g^2+h^2$ for points, planes, and lines, respectively; and we can therefore suppose that the tetrahedron $\eta_1\eta_2\eta_3\eta_4$ was originally taken as the tetrahedron of reference.

It will be convenient to begin by considering the transformation of line coordinates. Now, a condition that has to be satisfied is that $af+bg+ch$ shall be transformed into itself, and not into a multiple of itself.

This condition is not satisfied if we take the values of η_1, &c., just given; but it is satisfied if we take

$$\eta_1=\frac{1}{\sqrt{2}}(e_1+e_3),\quad \eta_2=\frac{i}{\sqrt{2}}(e_2+e_4),\quad \eta_3=\frac{-i}{\sqrt{2}}(e_1-e_3),$$

$$\eta_4=\frac{1}{\sqrt{2}}(e_2-e_4),$$

values which give

$$e_1=\frac{\eta_1+i\eta_3}{\sqrt{2}},\quad e_2=\frac{\eta_2+i\eta_4}{\sqrt{-2}},\quad e_3=\frac{\eta_1-i\eta_3}{\sqrt{2}},\quad e_4=\frac{\eta_2-i\eta_4}{\sqrt{-2}}.$$

We find

$$2\,(e_2e_3,\ e_3e_1,\ e_1e_2,\ e_1e_4,\ e_2e_4,\ e_3e_4)$$

$$= \left(\begin{array}{cccccc} -1, & 0, & i, & -1, & 0, & i \\ 0, & -2i, & 0, & 0, & 0, & 0 \\ -1, & 0, & -i, & 1, & 0, & i \\ -1, & 0, & -i, & -1, & 0, & -i \\ 0, & 0, & 0, & 0, & 2i, & 0 \\ 1, & 0, & -i, & -1, & 0, & i \end{array}\right)\!\!\!\!\bigg\langle\; \eta_2\eta_3,\ \eta_3\eta_1,\ \eta_1\eta_2,\ \eta_1\eta_4,\ \eta_2\eta_4,\ \eta_3\eta_4).$$

Now, let the coordinates of any line be $(abcfgh)$ with respect to $e_1e_2e_3e_4$, and $(ABCFGH)$ with respect to $\eta_1\eta_2\eta_3\eta_4$; then we must have

$$ae_2e_3+be_3e_1+ce_1e_2+fe_1e_4+ge_2e_4+he_3e_4$$
$$= A\eta_2\eta_3+B\eta_3\eta_1+C\eta_1\eta_2+F\eta_1\eta_4+G\eta_2\eta_4+H\eta_3\eta_4.$$

Substituting for e_2e_3, &c., their values in terms of $\eta_2\eta_3$, &c., and comparing coefficients of $\eta_2\,\eta_3$, &c., we get

$$2A = -(a+f)+(h-c),$$
$$2B = -2ib,$$
$$2C = i\left\{(a-f)-(h+c)\right\},$$
$$2F = -(a+f)-(h-c),$$
$$2G = 2ig,$$
$$2H = i\left\{(a-f)+(h+c)\right\};$$

and therefore

$$4\,(A^2+B^2+C^2+F^2+G^2+H^2)$$
$$= (a+f)^2+(h-c)^2-4b^2-(a-f)^2-(h+c)^2$$
$$+(a+f)^2+(h-c)^2-4g^2-(a-f)^2-(h+c)^2$$
$$= 8\,(af-ch)-4\,(b^2+g^2),$$

or $$A^2+B^2+C^2+F^2+G^2+H^2 = 2\,(af-ch)-(b^2+g^2).$$

It can be easily verified that we have

$$AF+BG+CH = af+bg+ch.$$

If we consider the transformation of point coordinates, we shall find, if $(xyzw)$ are the coordinates of a point with respect to $e_1e_2e_3e_4$, and

$(XYZW)$ are its coordinates referred to $\eta_1\eta_2\eta_3\eta_4$,

$$X^2+Y^2+Z^2+W^2 = 2(xz-yw).$$

IV.

We have now to consider the canonical form of the screw itself on which our transformations depend. I use ω as in my second note on the theory of screws, to denote the operation of taking the conjugate with respect to the absolute; so that, whatever x may be, ωx is its conjugate with respect to the absolute. The points of reference e_i were determined as the latent points of a certain matrix, which appertained to a certain screw; call this screw a, then the equation determining the latent points is

$$e_i\, a = \lambda_i\, \omega e_i.$$

Now, let $ABCFGH$ be the coordinates of a referred to the tetrahedron $e_1e_2e_3e_4$. Then we have

$$e_1 a = Ae_1e_2e_3 + Ge_1e_2e_4 + He_1e_3e_4.$$

But $$\omega e_1 = e_1e_2e_4.$$

Therefore $$G = \lambda_1,$$

$$A = H = 0.$$

Again, $$e_2 a = Be_2e_3e_1 + Fe_2e_1e_4,$$

$$\omega e_2 = -e_3e_2e_1 = e_2e_3e_1.$$

Therefore $$F = 0, \quad B = \lambda_2.$$

The equation $e_3 a = -\lambda_1 \omega e_3$ gives $C = 0$, and we get, as the canonical form, $$\lambda_2 e_3 e_1 + \lambda_1 e_2 e_4,$$

or $$iTa\left(\pm \sin\frac{\phi}{2}\, e_3e_1 \pm \cos\frac{\phi}{2}\, e_2e_4\right),$$

where there is nothing so far to determine the signs; but if we remember that we must have

$$2\lambda_1\lambda_2 = T^2 a \sin\phi,$$

it is obvious that the two signs must be different, and we can take

$$a = iTa\left(\sin\frac{\phi}{2}\, e_3e_1 - \cos\frac{\phi}{2}\, e_2e_4\right),*$$

* This transformation can also be effected by supposing the screw to be referred to $\eta_1\eta_2\eta_3\eta_4$, and then using the equations defining $e_1e_2e_3e_4$, and their expressions in terms of $\eta_1\eta_2\eta_3\eta_4$.

and we see at once that the axes of a are e_3e_1, e_2e_4. It should be noticed that we have now fixed the correspondence between the latent roots of Φ, the matrix defining the motion, and the points $e_1e_2e_3e_4$, viz., the order of the latent roots is

$$\frac{1+iTa\cos\frac{\phi}{2}}{1-iTa\cos\frac{\phi}{2}},\quad \frac{1-iTa\sin\frac{\phi}{2}}{1+iTa\sin\frac{\phi}{2}},\quad \frac{1-iTa\cos\frac{\phi}{2}}{1+iTa\cos\frac{\phi}{2}},\quad \frac{1+iTa\sin\frac{\phi}{2}}{1-iTa\sin\frac{\phi}{2}}.$$

I shall denote them as α, β, α^{-1}, β^{-1}.

V.

It will be worth while to give a few results connected with the transformations in the last two sections. I shall suppose that we take $e_1e_2e_3e_4$ as the tetrahedron of reference.

Let $a \equiv (abcfgh)$ be any screw; then we have

$$T^2a = 2(af-ch)-(b^2+g^2),$$

$$Saa' = af'+a'f-ch'-c'h-bb'-gg',$$

and therefore

$$\omega a \equiv (a,\ -g,\ -c,\ f,\ -b,\ -h),$$

$$2\xi a = (2a,\ b-g,\ 0,\ 2f,\ g-b,\ 0),$$

$$2\eta a = (0,\ b+g,\ 2c,\ 0,\ b+g,\ 2h),$$

$$\xi a = 0 \ \text{ if } \ a = f = b-g = 0,$$

$$\eta a = 0 \ \text{ if } \ c = h = b+g = 0.$$

Two screws a, a' are ξ-parallel if

$$\frac{a}{a'} = \frac{b-g}{b'-g'} = \frac{f}{f'};$$

they are η-parallel if

$$\frac{c}{c'} = \frac{b+g}{b'+g'} = \frac{h}{h'}.$$

The coordinates of a ξ-generator are

$$(0,\ \lambda,\ -1,\ 0,\ \lambda,\ \lambda^2).$$

The coordinates of an η-generator are

$$(1,\ \lambda,\ 0,\ \lambda^2,\ -\lambda,\ 0).$$

The equations of a ξ-generator are

$$z - \lambda y = 0,$$

$$w - \lambda x = 0.$$

The equations of an η-generator are

$$z - \lambda w = 0,$$

$$y - \lambda x = 0.$$

VI.

We have seen that the matrix Φ defining the motion can be reduced to the canonical form $\Phi = \dfrac{\alpha e_1, \beta e_2, \alpha^{-1} e_3, \beta^{-1} e_4}{e_1, \; e_2, \; e_3, \; e_4}$.

It follows that, if the coordinates of a point referred to the tetrahedron $e_1 e_2 e_3 e_4$ are $(xyzw)$, those of its new position referred to the same tetrahedron will be

$$\alpha x, \quad \beta y, \quad \alpha^{-1} z, \quad \beta^{-1} w.$$

Let P, P' be the two positions of the point. The distance between the points $(xyzw)$ and $(x'y'z'w')$ is given by

$$\cos PP' = \frac{xz' + x'z - yw' - y'w}{\sqrt{2(xz - yw)}\ \sqrt{2(x'z' - y'w')}},$$

and, therefore, for the two positions of P,

$$\cos PP' = (\alpha + \alpha^{-1}) \frac{xz}{2(xz - yw)} - (\beta + \beta^{-1}) \frac{yw}{2(xz - yw)}.$$

Now,
$$P e_3 e_1 = (x e_1 + y e_2 + z e_3 + w e_4)\, e_3 e_1$$
$$= -y e_3 e_2 e_1 + w e_3 e_1 e_4.$$

And, therefore,
$$T^2 (P e_3 e_1) = 2yw.*$$

Moreover, $T^2 (e_3 e_1) = -1$, and therefore, if θ is the distance of P from $e_3 e_1$, we have
$$\sin^2 \theta = -\frac{yw}{xz - yw},$$

and
$$\cos^2 \theta = \frac{xz}{xz - yw}.$$

---

* If $l e_2 e_3 e_4 + m e_3 e_1 e_4 + n e_1 e_2 e_4 + p e_3 e_2 e_1 \equiv l$ is any plane, we have, in the present system of coordinates, $T^2 l = 2(lm - np)$.

We have, therefore,

$$\cos PP' = \cos D_1 \cos^2\theta + \cos D_2 \sin^2\theta,$$

where
$$\cos D_1 = \frac{\alpha+\alpha^{-1}}{2},$$

$$\cos D_2 = \frac{\beta+\beta^{-1}}{2},$$

and $D_1 D_2$ are obviously the distances through which points on e_3e_1, e_2e_4 are moved. As regards these points, we see at once that points on either of these axes remain on the same axis. I proceed to find the simplest expressions for D_1, D_2. We have

$$\cos D_1 = \frac{\alpha+\alpha^{-1}}{2}$$

$$= \tfrac{1}{2}\frac{1+iTa\cos\frac{\phi}{2}}{1-iTa\cos\frac{\phi}{2}} + \tfrac{1}{2}\frac{1-iTa\cos\frac{\phi}{2}}{1+iTa\cos\frac{\phi}{2}} = \frac{1-T^2a\cos^2\frac{\phi}{2}}{1+T^2a\cos^2\frac{\phi}{2}},$$

and therefore
$$\tan\frac{D_1}{2} = Ta\cos\frac{\phi}{2}.$$

In the same way, we find

$$\tan\frac{D_2}{2} = Ta\sin\frac{\phi}{2}.$$

We have
$$\frac{\tan\frac{D_2}{2}}{\tan\frac{D_1}{2}} = \tan\frac{\phi}{2},$$

and $\tan\frac{\phi}{2}$ is what Sir Robert Ball calls the pitch of the screw.

We found

$$\cos PP' = \cos D = \cos D_1 \cos^2\theta + \sin D_1 \sin^2\theta.$$

Now, if P is taken on a fixed line, we have

$$\sin^2\theta = \sin^2\theta_1\cos^2\delta + \sin^2\theta_2\sin^2\delta,$$

$$\cos^2\theta = \cos^2\theta_1\cos^2\delta + \cos^2\theta_2\sin^2\delta,$$

if θ_1, θ_2 are the shortest distances between e_2e_3 and the line, and δ is the distance of P from the point where one of the perpendiculars cuts the line.

We have, therefore,

$$\cos D = \cos D_1 (\sin^2\theta \cos^2\delta + \sin^2\theta_2 \sin^2\delta)$$
$$+\cos D_2 (\cos^2\theta_1 \cos^2\delta + \cos^2\theta_2 \sin^2\delta)$$
$$= \cos\Delta_1 \cos^2\delta + \cos\Delta_2 \sin^2\delta,$$

where $$\cos\Delta_1 = \cos D_1 \sin^2\theta_1 + \cos D_2 \cos^2\theta_1,$$
$$\cos\Delta_2 = \cos D_1 \sin^2\theta_2 + \cos D_2 \cos^2\theta_2,$$

and Δ_1, Δ_2 are obviously the displacements of the points where the two shortest distances cut the line.

VII.

In this section I investigate the application of biquaternions to the representation of motions in any kind of space. In this application we have to consider a motion as a linear transformation of line coordinates; that is to say, we consider space as made up of screws.

I use the notations explained in my paper on biquaternions in Vol. VII. of the *American Journal of Mathematics.*

If a linear tranformation of line coordinates is to represent a motion, it must transform a line into a line, and it must leave the angle between two lines unaltered. It is obvious that if these conditions are to be satisfied, $af+bg+ch$ and $e^2(a^2+b^2+c^2)+f^2+g^2+h^2$ must be unaltered by the linear transformation. But if we represent the line by a bivector ρ, the two expressions just written are $\Omega N\rho\,/\,2$ and $\mho N\rho$ respectively, and we see that a linear transformation represents a line if, and only if, it leaves $N\rho$ unaltered. Now consider the bivector ϖ given by the equation

$$\varpi = Q\rho Q^{-1},$$

where Q is a biquaternion; ϖ is obviously got by operating on ρ with a certain linear transformation, and since all quaternion identities hold for biquaternions, we see that $N\varpi = N\rho$. It follows that the operator $Q(\ )\,Q^{-1}$, operating on a bivector, represents a motion. Moreover, we easily see that, if $\rho = VQ$, $\varpi = \rho$, and therefore the motion is specially related to the screw VQ.

In § 3 of the paper on biquaternions already referred to, I have shown how we can find the axis of a given biquaternion by dividing it by a certain biscalar. Now if we have $\varpi = Q\rho Q^{-1}$, it is obvious that we can divide Q by any biscalar without altering ϖ, and we can therefore suppose Q to be a special biquaternion. This remark simplifies the biquaternion formulæ, and is important as reducing the disposable constants in the value of ϖ from seven to six.

Now, if $Q = \Delta + \mathrm{A}$ is any quaternion, and P a vector, we have

$$\Pi = Q\mathrm{P}Q^{-1} = \frac{1}{NQ} Q\mathrm{P}KQ$$
$$= \frac{\mathrm{P}\,(\Delta^2 + \mathrm{A}^2) + 2\Delta V\mathrm{AP} - 2\mathrm{A}S\mathrm{AP}}{\Delta^2 - \mathrm{A}^2}.$$

We have to use this formula, remembering that all the quantities involved are biquaternions. I write

$$\Pi = \varpi + \omega\varpi', \quad \mathrm{P} = \rho + \omega\rho', \quad \Delta = \delta + \omega\delta', \quad \mathrm{A} = \alpha + \omega\alpha'.$$

I shall also suppose that Q is a special biquaternion. This condition gives

$$\delta\delta' = S\alpha\alpha',$$
$$NQ = \delta^2 + e^2\delta'^2 - \alpha^2 - e^2\alpha'^2.$$

Our object is to determine δ, δ' so that the motion represented by $Q\;(\;)\;Q^{-1}$ may be the same as that represented by the screw $\alpha + \omega\alpha'$ according to the principles used in the first part of this paper. To do this I suppose our coordinates chosen in such wise that

$$\alpha + \omega\alpha' = gj + b\omega j.$$

Then we must have* $a = c = f = h = 0$ in the value of Φ given in (1). I shall only consider one edge of the tetrahedron of reference.

I take $\mathrm{P} = i$; we get

$$NQ\,.\,\Pi = i\,(\delta^2 + \delta'^2 - g^2 - e^2b^2) - 2k\,(\delta g + e^2\delta' b) - 4bg\omega i - 2\,(\delta b + \delta' g)\,\omega k.$$

But, using the value of Φ given in (1), we get

$$\Delta\,.\,\Phi e_1 = (1 + e^2b^2)\,\{(1 - g^2)\,e_1 - 2ge_3\},$$
$$\Delta\,.\,\Phi e_4 = (1 + g^2)\,\{(1 - e^2b^2)\,e_4 - 2be_2\}.$$

And therefore, since

$$\Delta = (1 + g^2)(1 + e^2b^2),$$

$$\Delta\,.\,\Phi\,(e_1e_4) = -4bg\,e_2e_3 - 2b\,(1 - g^2)\,e_1e_2$$
$$+ (1 - g^2)(1 - e^2b^2)\,e_1e_4 - 2g\,(1 - e^2b^2)\,e_3e_4.$$

Now $$NQ = \delta^2 + \delta'^2 + g^2 + e^2b^2,$$

and $$e_2e_3 = \omega i, \quad e_1e_2 = \omega k, \quad e_1e_4 = i, \quad e_3e_4 = k.$$

---

* The screw $(abcfgh)$ is here represented by $(fi + gj + hk) + \omega\,(ai + bj + ck)$ and not as in my paper on biquaternions.

We see that the two expressions agree if we take

$$\delta = 1,$$

$$\delta' = -bg,$$

and therefore we have the theorem.

If $Q = 1+\alpha+\omega\,(S\alpha\alpha'+\alpha')$, then the operator $Q\;(\;)\;Q^{-1}$, operating on a bivector, represents the motion due to the screw $\alpha+\omega\alpha'$.

VIII.

In this and the following section I use the results of the last section to investigate formulæ for the angles and distances through which lines are moved, and for the composition of motions. The biquaternion Q will be supposed to be any special biquaternion, and will be taken as defining the motion. We have

$$\Pi\,(\Delta^2-\mathrm{A}^2) = \mathrm{P}\,(\Delta^2+\mathrm{A}^2)+2\Delta V\mathrm{AP}-2\mathrm{A}S\mathrm{AP}.$$

This gives

$$S\Pi\mathrm{P}\,.\,(\Delta^2-\mathrm{A}^2) = \mathrm{P}^2\,(\Delta^2+\mathrm{A}^2)-2S^2\mathrm{AP}.$$

This equation gives us expressions for the distance and angle through which P is moved. In the special case in which P is a line cutting the axis of A at right angles, we have

$$\Omega\mathrm{P}^2 = 0,$$

$$S\mathrm{AP} = 0,$$

and we get
$$\cos\,(\mathrm{P}\Pi) = \frac{\delta^2+e^2\delta'^2+a^2+e^2a'^2}{\delta^2+e^2\delta'^2-a^2-e^2a'^2},$$

and therefore

$$\cos^2\tfrac{1}{2}\,(\mathrm{P}\Pi) = \frac{\delta^2+e^2\delta'^2}{\delta^2+e^2\delta'^2-a^2-e^2a'^2}.$$

We have also
$$\mathrm{es}\,[\mathrm{P}\Pi] = \frac{4\delta\delta'}{\delta^2+e^2\delta'^2-a^2-e^2a'^2}.$$

It will be convenient to introduce the following notations: Q being any special biquaternion, let

$$\cos^2\,(Q) = \frac{\mho\,(SQ)^2}{NQ} = \frac{T^2\,(SQ)}{NQ},\text{*}$$

---

* I use the notation of my paper on biquaternions: $NQ = QKQ$, $T^2Q = \mho NQ$. It is convenient to write $\mho^2 Q$, $\Omega^2 Q$ for $(\mho Q)^2$, $(\Omega Q)^2$ respectively.

so that $$\sin^2(Q) = \frac{-\mho(VQ)^2}{NQ} = \frac{T^2VQ}{NQ},$$

and let $$\operatorname{es}\{Q\} = \frac{4\delta\delta'}{NQ} = \frac{2\Omega\,(SQ)^2}{NQ}.$$

We see, then, that for a line cutting VQ at right angles, we have

$$(\mathrm{P\Pi}) = 2\,(Q),$$

$$[\mathrm{P\Pi}] = \{Q\}.$$

It is not hard to get expressions for $(\mathrm{P\Pi})$ and $[\mathrm{P\Pi}]$ in the case of any screw P.

We have $$NQ\,.\,S\Pi\mathrm{P} = \mathrm{P}^2(\Delta^2+\mathrm{A}^2) - 2S^2\mathrm{AP},$$

which breaks up into

$$NQ\,.\,\mho S\Pi\mathrm{P} = \mho\mathrm{P}^2\,.\,\mho\,(\Delta^2+\mathrm{A}^2) + e^2\Omega\mathrm{P}^2\Omega\,(\Delta^2+\Delta^2)$$
$$-2\mho^2 S\mathrm{AP} - 2e^2\Omega^2 S\mathrm{AP},$$

$$NQ\,.\,\Omega S\Pi\mathrm{P} = \mho\mathrm{P}^2\,.\,\Omega\,(\Delta^2+\mathrm{A}^2) + e^2\Omega\mathrm{P}^2\mho\,(\Delta^2+\mathrm{A}^2) - 4\Omega S\mathrm{AP}\,.\,\mho S\mathrm{AP},$$

and therefore, since $T\Pi = T\mathrm{P}$, $S\mathrm{AP} = SQ\mathrm{P}$,

$$\cos(\Pi\mathrm{P}) = \cos 2\,(Q) + \sin^2 e\,[\mathrm{P}]\operatorname{es}\{Q\} - 2\cos^2(Q\mathrm{P}) - 2\sin^2 e\,[Q\mathrm{P}],$$

$$\operatorname{es}[\Pi\mathrm{P}] = \operatorname{es}\{Q\} + \sin^2 e\,[\mathrm{P}]\cos 2\,(Q) - 4\cos(Q\mathrm{P})\operatorname{es}[Q\mathrm{P}].$$

IX.

If we have $$\varpi = Q\rho Q^{-1},$$

we have $$Q'\varpi Q^{-1} = Q'Q\,.\,\rho\,(Q'Q)^{-1},$$

and therefore to compound two motions we multiply the corresponding biquaternions together.

It is important to prove that the product of two special biquaternions is a special biquaternion. A special biquaternion is defined by the equation

$$\Omega NQ = 0.$$

Now let Q, Q' be two biquaternions; we have

$$\Omega N(QQ') = \Omega NQNQ' = \Omega NQ\mho NQ' + \mho NQ\Omega NQ',$$

and therefore $\Omega N(QQ')$ vanishes if ΩNQ, $\Omega NQ'$ both vanish.

We have, if $Q'' = QQ'$,

$$\cos^2(Q'') = \frac{\mho\,SQQ' + e^2\Omega^2 SQQ'}{T^2QT^2Q'} = \cos^2(QKQ') + e^2\operatorname{es}^2[QKQ'].$$

Moreover,

$$\text{es}\{Q''\} = 4\frac{\mho SQQ'\Omega SQQ'}{T^2QT^2Q'} = 4\cos(QKQ')\,\text{es}\,[QKQ'].$$

If Q, Q' are given in the standard form, so that we have

$$Q = 1+\alpha+\omega\,(S\alpha\alpha'+\alpha') = 1+\omega S\alpha\alpha'+\mathrm{A},$$

$$Q' = 1+\alpha_1+\omega\,(S\alpha_1\alpha_1'+\alpha_1') = 1+\omega S\alpha_1\alpha_1'+\mathrm{A}_1,$$

we have

$$\mho SQQ' = 1+e^2S\alpha\alpha'S\alpha_1\alpha_1'+\mho S\mathrm{A}\mathrm{A}_1$$

$$= 1+\frac{e^2}{4}T^2\mathrm{A}T^2\mathrm{A}_1\,\text{es}\,[\mathrm{A}]\,\text{es}\,[\mathrm{A}_1]-T\mathrm{A}T\mathrm{A}_1\cos(\mathrm{A}\mathrm{A}_1),$$

$$\Omega SQQ' = S\alpha\alpha'+S\alpha_1\alpha_1'+\Omega S\mathrm{A}\mathrm{A}_1$$

$$= -\frac{T^2\mathrm{A}\,\text{es}\,[\mathrm{A}]}{2}-\frac{T^2\mathrm{A}_1'\,\text{es}\,[\mathrm{A}_1]}{2}-T\mathrm{A}T\mathrm{A}_1\,\text{es}\,[\mathrm{A}\mathrm{A}_1],$$

and we can, if we choose, substitute these values in the values of $\cos^2(Q'')$, es $\{Q''\}$ given above.

As regards the connexion of the angles (Q), $\{Q'\}$ with the pitch and tensor of the screw defining the motion, I add the following formulæ:—

$$\tan^2(Q) = \frac{T^2\mathrm{A}}{1+e^2S^2\alpha\alpha'}$$

$$= \frac{T^2\mathrm{A}}{1+\frac{e^2}{4}T^4\mathrm{A}\,\text{es}^2\,[\mathrm{A}]},$$

$$\text{es}\,\{Q\} = \frac{4S\alpha\alpha_1}{1+e^2S^2\alpha\alpha_1+T^2\mathrm{A}}$$

$$= -\frac{4T^2\mathrm{A}\,\text{es}\,[\mathrm{A}]}{(1+T^2\mathrm{A}\,\text{ec}^2\tfrac{1}{2}[\mathrm{A}])(1+e^2T^2\mathrm{A}\,\text{es}^2\tfrac{1}{2}[\mathrm{A}]}.$$

Lastly, I remark that, if $\mho SQQ'$ does not vanish, the screw resulting from two biquaternions is

$$\frac{VQQ'}{\mho SQQ'}.$$

On the Motion of a Liquid Ellipsoid under the Influence of its own Attraction. By A. B. Basset.

[*Read June* 10*th*, 1886.]

1. In the ninth volume of the *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Riemann has obtained equations for determining the motion of a mass of liquid, which rotates under the influence of its own attraction, in such a manner that its bounding surface always remains an ellipsoid with variable axes.

The motion of the liquid is supposed to be rotational, but the molecular rotation is assumed to be independent of the positions of individual particles of liquid, and the consequence of this assumption is, that the velocities at any point of the liquid are linear functions of the coordinates of that point. As regards their form, Riemann's equations leave nothing to be desired; but as the analysis by which he obtains them is somewhat complicated and difficult to follow, I propose in the present communication to deduce these equations by the dynamical method which Professor Greenhill has employed in his papers in the *Proceedings of the Cambridge Philosophical Society* (Vol. IV., pages 4 and 208), for dealing with the question of the steady motion of an ellipsoidal mass of liquid. It will be seen that the application of this method to the general case in which the axes are functions of the time, involves nothing more than the addition of the terms $\dot{a}x/a$, $\dot{b}y/b$, $\dot{c}z/c$ to the expressions for the component velocities obtained by Professor Greenhill; and also that, in differentiating with respect to the time, the axes of the ellipsoid must be regarded as functions of the time.

2. The motion of the liquid, as Professor Greenhill has pointed out, may be supposed to be generated by the two following operations, which are supposed to take place instantaneously one after the other.

1st, Let an ellipsoidal case, whose axes are a, b, c, be filled with liquid which is frozen, and then set in rotation with component angular velocities ξ, η, ζ about the principal axes.

2ndly, Let the liquid be melted, and additional angular velocities Ω_1, Ω_2, Ω_3 be impressed on the case.

If the axes vary with the time, we require the following third operation:—

Let the case be removed, and by means of a suitable impulsive pressure applied to the bounding surface, let the axes be made to vary with velocities $\dot{a}$, $\dot{b}$, $\dot{c}$.

Let x, y, z be the coordinates of a particle of liquid referred to the principal axes; u, v, w the component velocities of the particle; and U, V, W the component velocities relative to the axes; also let ω_1, ω_2, ω_3 be the angular velocities of the axes, so that

$$\omega_1 = \Omega_1 + \xi, \quad \omega_2 = \Omega_2 + \eta, \quad \omega_3 = \Omega_3 + \zeta.$$

The boundary condition is

$$\frac{dF}{dt} + U\frac{dF}{dx} + V\frac{dF}{dy} + W\frac{dF}{dz} \quad \text{......................} (1),$$

where $$F = (x/a)^2 + (y/b)^2 + (z/c)^2 - 1 = 0,$$

and $$U = u + \omega_3 y - \omega_2 z, \text{ \&c., \&c.}$$

Equation (1) can be satisfied by assuming

$$u = l_1 x + m_1 y + n_1 z,$$
$$v = l_2 x + m_2 y + n_2 z,$$
$$w = l_3 x + m_3 y + n_3 z,$$

where l_1, m_1, &c., are independent of x, y, and z. Substituting in (1), and equating coefficients of powers and products of x, y, z to zero, we obtain

$$l_1 = \dot{a}/a, \quad m_2 = \dot{b}/b, \quad n_3 = \dot{c}/c,$$
$$(n_2 + \omega_1)\, c^2 + (m_3 - \omega_1)\, b^2 = 0,$$
$$(l_3 + \omega_2)\, a^2 + (n_1 - \omega_2)\, c^2 = 0,$$
$$(m_1 + \omega_3)\, b^2 + (l_2 - \omega_3)\, a^2 = 0.$$

But, from the mode of generation, ξ, η, ζ are independent of x, y, and z;

therefore $$2\xi = m_3 - n_2, \quad 2\eta = n_1 - l_3, \quad 2\zeta = l_2 - m_1.$$

Hence the nine coefficients are completely determined, and we shall finally obtain

$$u = \frac{\dot{a}x}{a} + \frac{\omega_1(a^2 - b^2) - 2a^2\zeta}{a^2 + b^2}\, y + \frac{\omega_2(c^2 - a^2) + 2a^2\eta}{c^2 + a^2}\, z \quad \text{............} (2),$$

with symmetrical expressions for v and w.

These values of u, v, and w obviously satisfy the equation of continuity, since on account of the constancy of volume

$$\dot{a}/a + \dot{b}/b + \dot{c}/c = 0.$$

The general equations for the pressure referred to moving axes are*

$$\frac{1}{\rho}\frac{dp}{dx} - X + \frac{du}{dt} - v\omega_3 + w\omega_2 + U\frac{du}{dx} + V\frac{du}{dy} + W\frac{du}{dz} = 0 \quad \ldots\ldots(3),$$

&c. &c.,

and by eliminating the pressure and potential, the equations for molecular rotation are found to be

$$\frac{d\xi}{dt} - \eta\omega_3 + \zeta\omega_2 + U\frac{d\xi}{dx} + V\frac{d\xi}{dy} + W\frac{d\xi}{dz} = \xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz} \ldots\ldots(4),$$

&c. &c.

Substituting the values of u, v, and w, from (2) in (4), we shall obtain

$$\frac{d}{dt}\left(\frac{\xi}{a}\right) - \frac{2ab}{a^2+b^2}\Omega_3\left(\frac{\eta}{b}\right) + \frac{2ca}{c^2+a^2}\Omega_2\left(\frac{\zeta}{c}\right) = 0 \ldots\ldots\ldots\ldots(5),$$

&c. &c.

If h_1, h_2, h_3 be the components of angular momentum, then

$$h_1 = \frac{M}{5(b^2+c^2)}\left\{(b^2-c^2)^2\omega_1 + 4b^2c^2\xi\right\} \quad \ldots\ldots\ldots\ldots\ldots(6),$$

&c. &c.

$$\frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

&c. &c.

where M is the mass of the liquid.

In order to facilitate the calculation, Riemann introduces six new quantities u, v, w, u', v', w', such that

$$\left.\begin{aligned} u+u' &= \omega_1, & v+v' &= \omega_2, & w+w' &= \omega_3 \\ u-u' &= \frac{2bc\Omega_1}{b^2+c^2}, & v-v' &= \frac{2ca\Omega_2}{c^2+a^2}, & w-w' &= \frac{2ab_3\Omega}{a^2+b^2} \end{aligned}\right\} \ldots\ldots(8).$$

---

* Equation (3) may be shortly proved by remembering that $X - \frac{1}{\rho}\frac{dp}{dx}$ = the acceleration of a particle of liquid parallel to the axis of x. Now, if $u+\delta u$ be the velocity at time $t+\delta t$ parallel to the new position of the axis of x, of the particle whose coordinates at time t are x, y, z, then, since

$$u = f(x, y, z, t), \qquad u+\delta u = f(x+U\delta t,\ y+V\delta t,\ z+W\delta t,\ t+\delta t),$$

therefore
$$\frac{\delta u}{\delta t} = \frac{du}{dt} + U\frac{du}{dx} + V\frac{du}{dy} + W\frac{du}{dz},$$

and the acceleration $= \frac{\delta u}{\delta t} - v\omega_3 + w\omega_2$.

$$\left.\begin{aligned} \text{Whence} \qquad \xi &= \frac{(b+c)^2 u' - (b-c)^2 u}{2bc}, \text{ \&c. \&c.} \\ h_1 &= \frac{M}{5}\left\{(b+c)^2 u' + (b-c)^2 u\right\}, \text{ \&c. \&c.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(9).$$

Substituting these values of ξ, η, ζ and h_1, h_2, h_3 in (5) and (7), and then multiplying (5) by $2Mabc/5$ and adding to (7), we obtain

$$(b+c)\frac{du'}{dt} + 2u'\frac{d}{dt}(b+c) + (b-c+2a)\,vw' + (b-c-2a)\,v'w = 0 \ldots(10).$$

Similarly, by subtraction, we obtain

$$(b-c)\frac{du}{dt} + 2u\frac{d}{dt}(b-c) + (b+c-2a)\,vw + (b+c+2a)\,v'w' = 0 \ldots(11).$$

Four other equations can respectively be written down by symmetry, and we thus obtain six equations of motion. The three remaining equations can be obtained as follows. The potential of the liquid at an internal point is

$$V = \tfrac{1}{2}(Ax^2 + By^2 + Cz^2) - H,$$

where $$H = \frac{3M}{4}\int_0^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

and $$A = -\frac{2}{a}\frac{dH}{da}, \text{ \&c., \&c.}$$

Now, Mr. H. W. G. Mackenzie has shown very shortly, at the end of Prof. Greenhill's first paper, that the equations determining the pressure may be reduced to the form

$$\frac{1}{\rho}\frac{dp}{dx} + (A+\alpha)\,x = 0,$$

$$\frac{1}{\rho}\frac{dp}{dy} + (B+\beta)\,y = 0,$$

$$\frac{1}{\rho}\frac{dp}{dz} + (C+\gamma)\,z = 0,$$

where α, β, and γ are quantities independent of x, y, and z, and which will be hereafter determined. Integrating, we obtain

$$\frac{p}{\rho} + \Pi + \tfrac{1}{2}\left\{(A+\alpha)\,x^2 + (B+\beta)\,y^2 + (C+\gamma)\,z^2\right\} = 0 \ldots\ldots(12).$$

Since the external surface is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, we must have $(A+\alpha)\,a^2 = (B+\beta)\,b^2 = (C+\gamma)\,c^2 = 2\sigma$ ....(13), where σ is a function of the time.

Hence (12) may be written

$$\frac{p}{\rho} = \varpi + \frac{\sigma}{\rho}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \quad \text{..............(14).}$$

In order that the external surface may be a free surface, it is necessary that ϖ should vanish, and consequently σ must never become negative.

Returning to equation (3), we see that α is the coefficient of x in the expression for the component acceleration parallel to x of a liquid particle, and therefore

$$\alpha = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) - \frac{w+w'}{a}\{(a-b)\,w + (a+b)\,w'\}$$
$$+ \frac{v+v'}{a}\{(c-a)\,v - (c+a)\,v'\}$$
$$= \frac{1}{a}\frac{d^2a}{dt^2} - \frac{2}{a}(a-b)\,w^2 - \frac{2}{a}(a+b)\,w'^2 - \frac{2}{a}(a-c)\,v^2 - \frac{2}{a}(a+c)\,v'^2.$$

Whence, by (13),

$$\frac{1}{2}\frac{d^2a}{dt^2} - (a-c)\,v^2 - (a+c)\,v'^2 - (a-b)\,w^2 - (a+b)\,w'^2 = \frac{\sigma}{a} - \frac{Aa}{2} \quad \text{...........(15).}$$

Two other symmetrical equations can be obtained; hence, collecting our results, we have the following ten equations:

$$\left.\begin{aligned}
&\tfrac{1}{2}\ddot{a} - (a-c)\,v^2 - (a+c)\,v'^2 - (a-b)\,w^2 - (a+b)\,w'^2 = \frac{\sigma}{a} - \frac{Aa}{2}\\
&\tfrac{1}{2}\ddot{b} - (b-a)\,w^2 - (b+a)\,w'^2 - (b-c)\,u^2 - (b+c)\,u'^2 = \frac{\sigma}{b} - \frac{Bb}{2}\\
&\tfrac{1}{2}\ddot{c} - (c-b)\,u^2 - (c+b)\,u'^2 - (c-a)\,v^2 - (c+a)\,v'^2 = \frac{\sigma}{c} - \frac{Cc}{2}\\
&(b-c)\,\dot{u} + 2u\,(\dot{b}-\dot{c}) + (b+c-2a)\,vw + (b+c+2a)\,v'w' = 0\\
&(b+c)\,\dot{u}' + 2u'\,(\dot{b}+\dot{c}) + (b-c+2a)\,vw' + (b-c-2a)\,v'w = 0\\
&(c-a)\,\dot{v} + 2v\,(\dot{c}-\dot{a}) + (c+a-2b)\,wu + (c+a+2b)\,w'u' = 0\\
&(c+a)\,\dot{v}' + 2v'\,(\dot{c}+\dot{a}) + (c-a+2b)\,wu' + (c-a-2b)\,w'u = 0\\
&(a-b)\,\dot{w} + 2w\,(\dot{a}-\dot{b}) + (a+b-2c)\,uv + (a+b+2c)\,u'v' = 0\\
&(a+b)\,\dot{w}' + 2w'\,(\dot{a}+\dot{b}) + (a-b+2c)\,uv' + (a-b-2c)\,u'v = 0\\
&\qquad\qquad abc = \text{const.}
\end{aligned}\right\} \text{..(16).}$$

These are Riemann's equations of motion. They furnish ten independent relations between the ten unknown quantities a, b, c, ω_1, ω_2, ω_3, ξ, η, ζ, and σ, and are therefore sufficient for the solution of the problem.

3. Three first integrals of the general equations (16) can be at once obtained. Multiply equations (5) by ξ/a, η/b, ζ/c respectively, and add, and we obtain

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2} = \text{const.} \qquad (17).$$

The second integral is

$$h_1^2+h_2^2+h_3^2 = \text{const.} \qquad (18),$$

which expresses the fact that the angular momentum is constant.

The third integral is the equation of energy

$$T+U = \text{const.} \qquad (19).$$

Since $$\rho\iiint x^2\,dx\,dy\,dz = \frac{Ma^2}{5},$$

and $$\iiint xy\,dx\,dy\,dz = 0,$$

we obtain, from (2),

$$T = \frac{M}{10}\Big[\dot{a}^2+\dot{b}^2+\dot{c}^2+\frac{\omega_1^2(b^2-c^2)^2}{b^2+c^2}+\frac{\omega_2^2(c^2-a^2)^2}{c^2+a^2}+\frac{\omega_3^2(a^2-b^2)^2}{a^2+b^2}$$
$$+\frac{4b^2c^2\xi^2}{b^2+c^2}+\frac{4c^2a^2\eta^2}{c^2+a^2}+\frac{4a^2b^2\zeta^2}{a^2+b^2}\Big] \qquad (20).$$

Now $$U = \tfrac{1}{2}\rho\iiint V\,dx\,dy\,dz^*$$

$$= \frac{3M^2}{8}\int_0^\infty\Big[\frac{1}{5}\Big(\frac{a^2}{a^2+\lambda}+\frac{b^2}{b^2+\lambda}+\frac{c^2}{c^2+\lambda}\Big)-1\Big]\frac{d\lambda}{P},$$

therefore $$U = -\frac{3M^2}{20}\int_0^\infty\frac{d\lambda}{P}+\frac{3M^2}{20}\int_0^\infty\lambda\,\frac{d}{d\lambda}\Big(\frac{1}{P}\Big)\,d\lambda,$$

where $$P = \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}.$$

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* Maxwell's *Electricity*, Vol. I., Art. 85.

Integrating the last integral by parts, we obtain

$$U = -\frac{2M\pi\rho abc}{5}\int_0^\infty \frac{d\lambda}{P} \quad\text{..........................(21).}$$

4. Dirichlet's equations for the oscillations of a spheroid may be deduced by the preceding method.* Let the density of the spheroid be unity, and let ω_1, ω_2, ξ, η, Ω_1, Ω_2, Ω_3 be each zero; also let $a = b$, $\omega_3 = \zeta$; so that $u = u' = \zeta/2$.

From the last of equations (5), we obtain

$$\frac{d}{dt}\left(\frac{\zeta}{c}\right) = 0,$$

therefore
$$\frac{\zeta}{c} = \frac{\zeta_0}{c_0},$$

where the suffixes denote the initial values of the quantities.

Let $D^3 = a^2c$, and let us introduce two new variables α and ρ, such that
$$\alpha = D^2/a^2 = c/D;$$

and
$$\rho = \zeta/(2\pi)^{\frac{1}{2}} = \zeta_0 c/c_0\sqrt{2\pi} = \rho_0\alpha/\alpha_0.$$

From the first and third of equations (16), we obtain

$$-\frac{\ddot{\alpha}}{2} + \frac{3\dot{\alpha}^2}{4\alpha} - 2\pi\rho^2\alpha = \frac{2\sigma\alpha^2}{D^2} - A\alpha,$$

$$\frac{\ddot{\alpha}}{2} = \frac{\sigma}{D^2\alpha} - \frac{C\alpha}{2}.$$

Eliminating $\ddot{\alpha}$ and σ, remembering that $A + C/2 = 2\pi$, we obtain

$$\frac{\sigma}{D^2}\left(2\alpha + \frac{1}{\alpha^2}\right) = 2\pi\,(1-\rho^2) + \frac{3\dot{\alpha}^2}{4\alpha^2} \quad\text{..................(22),}$$

$$2\left(2 + \frac{1}{\alpha^3}\right)\ddot{\alpha} - \frac{3\dot{\alpha}^2}{4\alpha^4} + 8\pi\left(\frac{\rho_0}{\alpha_0}\right)^2 = 4\left(\frac{A}{\alpha^3} - C\alpha\right) \quad\text{.........(23).}$$

If we put
$$f(\alpha) = \int_0^\infty \frac{ds}{(1+\alpha s)\left(1+\frac{s}{\alpha^2}\right)^{\frac{1}{2}}},$$

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* *Crelle*, Vol. LVIII., p. 209.

the left-hand side of the last equation can be easily shown to be equal to $8\pi f'(a)$. Multiplying by $\dot{a}$ and integrating, we obtain

$$\left(2+\frac{1}{a^3}\right)\dot{a}^2+8\pi\left\{\left(\frac{\rho_0}{a_0}\right)^2 a-f(a)\right\} = \text{const.} \quad \text{.........(24)},$$

which is the equation of energy.

Equations (22), (23), and (24) are the equations obtained by Dirichlet.

Solution of the Cubic and Quartic Equations by means of Weierstrass's Elliptic Functions. By A. G. Greenhill.

[*Read May* 13*th*, 1886.]

A. *Solution of the Cubic Equation.*

1. The solution of the cubic equation, when presented in the form

$$4x^3-Sx-T=0,$$

by means of the trigonometrical circular functions, is well known; for, putting $x = ny$, then

$$4y^3-\frac{S}{n^2}y-\frac{T}{n^3}=0,$$

and, comparing this equation with

$$4\cos^3 a-3\cos a-\cos 3a = 0,$$

we can put $\quad y = \cos a, \quad$ and $\quad x = n\cos a,$

provided that $\quad n^2 = \tfrac{1}{3}S, \quad$ and $\quad \cos 3a = \dfrac{T}{n^3};$

the other two roots being $n\cos(a \pm \tfrac{2}{3}\pi)$.

Denoting the *discriminant* S^3-27T^2 by Δ, and the *absolute invariant* $\dfrac{S^3}{\Delta}$ by J, according to Klein, then

$$\cos^2 3a = \frac{T^2}{n^6} = \frac{27T^2}{S^3} = \frac{J-1}{J},$$

$$\sin^2 3a = \frac{1}{J}, \quad \text{or} \quad \operatorname{cosec}^2 3a = J.$$

2. Changing the sign of x, the cubic equation becomes

$$4x^3 - Sx + T = 0,$$

which may be compared with

$$4\sin^3 a - 3\sin a + \sin 3a = 0,$$

so that $$x = n\sin a, \quad \text{or} \quad n\sin(a \pm \tfrac{2}{3}\pi),$$

provided that $$n^2 = \tfrac{1}{3}S, \text{ and } \sin 3a = \frac{T}{n^3};$$

and then $$\sin^2 3a = \frac{T^2}{n^6} = \frac{27T^2}{S^3} = \frac{J-1}{J},$$

$$\cos^2 3a = \frac{1}{J}, \quad \text{or} \quad \sec^2 3a = J.$$

In these two cases it is assumed that

$$\Delta = S^3 - 27T^2$$

is positive, so that all three roots of the cubic equation are real.

3. But, if Δ is negative, two of the roots of the cubic are imaginary; and, if S is positive, the equation

$$4x^3 - Sx - T = 0$$

must be compared with

$$4\cosh^3 a - 3\cosh a - \cosh 3a = 0,$$

and then the roots of the equation are

$$n\cosh a \quad \text{and} \quad n\cosh(a \pm \tfrac{2}{3}\pi i),$$

provided that $$n^2 = \tfrac{1}{3}S, \text{ and } \cosh 3a = \frac{T}{n^3};$$

so that $$\cosh^2 3a = \frac{T^2}{n^6} = \frac{27T^2}{S^3} = \frac{J-1}{J},$$

$$\sinh^2 3a = -\frac{1}{J}, \text{ or } \operatorname{cosech}^2 3a = -J;$$

and these are real, because J is negative.

4. If Δ is negative, and S is negative, then, changing the sign of S, the cubic equation $$4x^3 + Sx - T = 0$$

must be compared with

$$4\sinh^3 a + 3\sinh a - \sinh 3a = 0,$$

and then the roots of the equation are

$$n\sinh a \quad \text{and} \quad n\sinh(a \pm \tfrac{2}{3}\pi i),$$

provided that $\qquad n^2 = \tfrac{1}{3}S, \quad \text{and} \quad \sinh 3a = \dfrac{T}{n^3};$

and then $\qquad \sinh^2 3a = \dfrac{T^2}{n^6} = \dfrac{27T^2}{S^3},$

$$\cosh^2 3a = \frac{S^3 + 27T^2}{S^3} = \frac{1}{J}, \text{ or } \operatorname{sech}^2 3a = J.$$

Similarly, changing the sign of x, the roots of the cubic equation

$$4x^3 - Sx + T = 0$$

will be $\qquad -n\cosh a \quad \text{and} \quad -n\cosh(a \pm \tfrac{2}{3}\pi i),$

where $\qquad \operatorname{cosech}^2 3a = -J,$

J being negative; and the roots of the cubic equation

$$4x^3 + Sx + T = 0$$

will be $\qquad -n\sinh a \quad \text{and} \quad -n\sinh(a \pm \tfrac{2}{3}\pi i),$

where $\qquad \operatorname{sech}^2 3a = J.$

According to this method, the solution of the cubic, when only one root is real, depends on the values of the *hyperbolic* functions, which have the inconvenience of an infinite period, and so cannot conveniently be tabulated.

5. In the preceding cubics the second term of the equation has been removed; but, if we consider their reciprocal equations, we shall have a cubic equation of the form

$$z^3 + az^2 - 4b = 0,$$

a cubic equation with the *third* term removed, equivalent to the preceding equation $\qquad 4x^3 - Sx - T = 0,$

if $\qquad z = \dfrac{1}{x}, \quad a = \dfrac{S}{T}, \quad b = \dfrac{1}{T};$

and the roots of this new cubic in z will be all real, or one real and

two imaginary as before, according as

$$\Delta = S^3 - 27T^2 \text{ is positive or negative,}$$

$$= \frac{a^3 - 27b}{b^3} \qquad ,, \qquad ,,$$

We may always suppose T, and therefore b, is positive; for, if negative, changing the sign of x and z would make them positive in the equations; the roots of the new equation

$$z^3 + az^2 - 4b = 0$$

will therefore be all real, or one real and two imaginary, as $a^3 - 27b$ is positive or negative, on the supposition that T and b are positive.

6. Now, consider two variable quantities s and t, connected by the

relation $$t = s - \frac{g_3}{s^2};$$

then $$\frac{dt}{ds} = 1 + \frac{2g_3}{s^3},$$

and $$4t^3 + h_3 = (4s^3 - g_3)\left(1 + \frac{2g_3}{s^3}\right)^2,$$

if $$h_3 = 27g_3;$$

so that $$\frac{dt}{\sqrt{(4t^3 + h_3)}} = \frac{ds}{\sqrt{(4s^3 - g_3)}},$$

and $$\int_t^\infty \frac{dt}{\sqrt{(4t^3 + h_3)}} = \int_s^\infty \frac{ds}{\sqrt{(4s^3 - g_3)}} = u \text{ suppose.}$$

But, according to the definitions of Weierstrass, the absolutely simplest elliptic function, denoted by pu, of a variable quantity u, is

defined by $$u = \int_s^\infty \frac{ds}{\sqrt{(4s^3 - g_2 s - g_3)}},$$

and $$s = \mathrm{p}\, u,$$

so that $$\frac{ds}{du} = \mathrm{p}' u = -\sqrt{(4s^3 - g_2 s - g_3)}.$$

When it is desirable to indicate the quantities g_2 and g_3, called the *invariants*, then the notation

$$s = \mathrm{p}\,(u;\ g_2, g_3)$$

is employed; so that we may now write

$$s = \wp(u;\ 0,\ g_3),$$
$$t = \wp(u;\ 0,\ -h_3).$$

7. By means of the fundamental relation

$$\wp(u;\ g_2, g_3) = m^2\wp\left(mu;\ \frac{g_2}{m^4},\ \frac{g_3}{m^6}\right),$$

we find, putting $m^6 = -27$, $m^3 = -3$, $m = i\sqrt{3}$, that

$$t = \wp(u;\ 0, -h_3) = -3\wp(iu\sqrt{3};\ 0, g_3),$$

since $$h_3 = 27g_3;$$

and therefore, omitting the indication of g_3,

$$-3\wp(iu\sqrt{3}) = \wp u - \frac{g_3}{\wp^2 u},$$

or $$\wp^3 u + 3\wp(iu\sqrt{3})\,\wp^2 u - g_3 = 0;$$

and, comparing this with the equation

$$z^3 + az^2 - 4b = 0,$$

we have $$z = \wp u,$$

provided that $$a = 3\wp(iu\sqrt{3}),$$
$$g_3 = 4b, \quad g_2 = 0.$$

8. In order to tabulate the function $\wp u$, we must select some particular value of g_3; we shall find it convenient to put $g_3 = 4$, and then,

if $s = \wp u$, $$u = \int_s^\infty \frac{ds}{\sqrt{(4s^3-4)}},$$

or $$2u = \int_s^\infty \frac{ds}{\sqrt{(s^3-1)}};$$

and then, in the notation of Legendre and Jacobi,

$$s = \wp u = 1 + \sqrt{3}\,\frac{1+\operatorname{cn} 2u\sqrt[4]{3}}{1-\operatorname{cn} 2u\sqrt[4]{3}}, \text{ for } k = \sin 15°.$$

9. In the general notation of Weierstrass, the roots of the equation

$$4s^3 - g_2 s - g_3 = 0$$

are denoted by e_1, e_2, e_3; so that

$$4s^3 - g_2 s - g_3 = 4(s-e_1)(s-e_2)(s-e_3);$$

and, if ω_1, ω_2, ω_3 denote corresponding values of u, so that

$$\wp\,\omega_1 = e_1, \quad \wp\,\omega_2 = e_2, \quad \wp\,\omega_3 = e_3;$$

then ω_1, ω_2, ω_3 are called the *periods* of the elliptic functions; but they are connected by the relation

$$\omega_1 + \omega_2 + \omega_3 = 0,$$

also

$$e_1 + e_2 + e_3 = 0.$$

In our case of $g_2 = 0$, two of the quantities e_1, e_2, e_3 are imaginary; e_2 is then taken to be real, and, with $g_3 = 4$, we have

$$e_1 = \omega, \quad e_2 = 1, \quad e_3 = \omega^2;$$

ω and ω^2 denoting the imaginary cube roots of unity, such that

$$\omega - \omega^2 = i\sqrt{3}.$$

Then, since $s = 1$ when $u = \omega_2$, and $2u\sqrt[4]{3} = 2K$, in Jacobi's notation; therefore $K = \omega_2\sqrt[4]{3}$.

Also $\omega_3 - \omega_1$ is positive imaginary, and is denoted by ω_2' by Schwarz, and then $iK' = \omega_2'\sqrt[4]{3}$; so that

$$\frac{\omega_2'}{\omega_2} = i\,\frac{K'}{K} = i\sqrt{3}.$$

10. Then, as u decreases from ω_2 to 0, $\wp u$ will pass through all real values from 1 to ∞; and as iu increases from 0 to $\omega_2 i$, or $iu\sqrt{3}$ from 0 to ω_2', $\wp(iu\sqrt{3})$ will pass through all real values from $-\infty$ through 0 to $+1$; as exhibited in the following Table, kindly calculated for me by Mr. A. G. Hadcock, Inspector of Ordnance Machinery, Royal Artillery, in which the periods ω_2 and ω_2' have each been divided into 180 equal parts, and the corresponding values of $\wp u$ tabulated in the same horizontal line.

Then $\omega_2 = 1{\cdot}2143$, and $\wp\dfrac{r\omega_2}{180}$ can be calculated from the formula

$$\wp u = \frac{1}{u^2} - \frac{u^4}{7} + \frac{u^{10}}{7^2 . 13} \cdots\cdots.$$

Also, denoting $\wp\dfrac{r\omega_2}{180}$ by s and $\wp\dfrac{r\omega_2'}{180}$ by S, then

$$S = -\frac{1}{3}\left(s - \frac{4}{s^2}\right),$$

whence $\wp\dfrac{r\omega_2'}{180}$ can be calculated when $\wp\dfrac{r\omega_2}{180}$ is known.

	$s = \mathrm{p}\,\frac{r\omega_2}{180}$	$S = \mathrm{p}\,\frac{r\omega_2'}{180}$		$s = \mathrm{p}\,\frac{r\omega_2}{180}$	$S = \mathrm{p}\,\frac{r\omega_2'}{180}$
$r =$ 0	+ Infinity	− Infinity			
1	+ 21972·6	− 7324·200	$r =$ 46	+ 10·3851	− 3·44934
2	5494·39	1831·463	47	9·94817	3·30258
3	2331·61	777·2033	48	9·53820	3·16474
4	1373·21	457·7366	49	9·15299	3·03507
5	878·805	292·9350	50	8·79071	2·91297
6	599·074	199·6913	51	8·44971	2·79790
7	448·413	149·4710	52	8·12796	2·68914
8	343·303	114·4343	53	7·82443	2·59636
9	271·277	90·4256	54	7·53768	2·48909
10	219·726	73·2430	55	7·26630	2·39685
11	181·586	60·5286	56	7·00940	2·30932
12	152·593	50·8643	57	6·76574	2·22611
13	130·016	43·3385	58	6·53496	2·14710
14	112·103	37·3675	59	6·31568	2·07179
15	97·6523	32·55062	60	6·10721	2·00000
16	85·8317	28·61038	61	5·90906	1·93149
17	76·0292	25·34283	62	5·72044	1·86606
18	67·8150	22·60471	63	5·54062	1·80344
19	60·8671	20·28867	64	5·36929	1·74351
20	54·9316	18·31009	65	5·20585	1·68608
21	49·8237	16·60736	66	5·04976	1·63096
22	45·3966	15·13155	67	4·90066	1·57803
23	41·5363	13·84466	68	4·75815	1·52716
24	38·1466	12·71461	69	4·62180	1·47818
25	35·1554	11·71738	70	4·49772	1·43333
26	32·5043	10·83350	71	4·36622	1·38546
27	30·1407	10·04543	72	4·24650	1·34156
28	28·0260	9·34030	73	4·13150	1·29905
29	26·1262	8·70678	74	4·02134	1·25799
30	24·4142	8·13582	75	3·91554	1·21821
31	22·8643	7·61888	76	3·81391	1·17964
32	21·4574	7·14956	77	3·71629	1·14222
33	20·1766	6·72226	78	3·62241	1·10586
34	19·0077	6·33221	79	3·53214	1·07051
35	17·9369	5·97482	80	3·44533	1·03612
36	16·9542	5·64676	81	3·36163	1·00255
37	16·0506	5·34502	82	3·28113	·96986
38	15·2169	5·06654	83	3·20353	·93792
39	14·4465	4·80911	84	3·12871	·90669
40	13·7332	4·57066	85	3·05658	·87615
41	13·0719	4·34950	86	2·98703	·84623
42	12·4569	4·14371	87	2·91989	·81690
43	11·8843	3·95199	88	2·85504	·78811
44	11·3505	3·77316	89	2·79248	·75984
45	+ 10·8517	− 3·60591	90	+ 2·73200	− ·73202

	$s = \mathrm{p}\,\frac{r\omega_2}{180}$	$S = \mathrm{p}\,\frac{r\omega_2'}{180}$		$s = \mathrm{p}\,\frac{r\omega_2}{180}$	$S = \mathrm{p}\,\frac{r\omega_2'}{180}$
$r=91$	+ 2·67366	− ·70470	$r=136$	+ 1·28989	+ ·37141
92	2·61713	·67770	137	1·27562	·39420
93	2·56253	·65112	138	1·26183	·41679
94	2·50974	·62490	139	1·24846	·43929
95	2·45868	·59900	140	1·23564	·46140
96	2·40928	·57339	141	1·22310	·48357
97	2·36141	·54802	142	1·21105	·50543
98	2·31541	·52310	143	1·19934	·52716
99	2·27023	·49804	144	1·18798	·54877
100	2·22679	·47337	145	1·17710	·56994
101	2·18467	·44886	146	1·16657	·59090
102	2·14303	·42402	147	1·15639	·61163
103	2·10347	·39980	148	1·14662	·63195
104	2·06523	·37580	149	1·13720	·65196
105	2·02809	·35187	150	1·12805	·67180
106	1·99207	·32803	151	1·11932	·69112
107	1·95715	·30429	152	1·11101	·70987
108	1·92327	·28063	153	1·10290	·72851
109	1·89043	·25705	154	1·09514	·74669
110	1·85856	·23352	155	1·08773	·76436
111	1·82745	·20990	156	1·08073	·78133
112	1·79759	·18656	157	1·07394	·79807
113	1·76849	·16317	158	1·06757	·81405
114	1·74016	·13974	159	1·06147	·82956
115	1·71272	·11637	160	1·05558	·84476
116	1·68605	·09298	161	1·05011	·85909
117	1·66069	·07010	162	1·04484	·87306
118	1·63554	·04674	163	1·03999	·88610
119	1·61108	·02333	164	1·03528	·89891
120	1·58746	− ·00000	165	1·03099	·91073
121	1·56439	+ ·02336	166	1·02690	·92210
122	1·54194	·04681	167	1·02323	·93242
123	1·52275	·06744	168	1·01969	·94244
124	1·49926	·09343	169	1·01651	·95155
125	1·47876	·11682	170	1·01360	·95994
126	1·45887	·14019	171	1·01103	·96738
127	1·43961	·16348	172	1·00861	·97447
128	1·42091	·18677	173	1·00653	·98058
129	1·40275	·21002	174	1·00473	·98590
130	1·38509	·23331	175	1·00327	·99023
131	1·36797	·25651	176	1·00229	·99340
132	1·35141	·27959	177	1·00120	·99620
133	1·33527	·30273	178	1·00050	·99840
134	1·31961	·32580	179	1·00010	·99960
135	+ 1·30458	+ ·34856	180	+ 1·00000	+ 1·00000

11. Now, put $z = \dfrac{y}{m}$ in the equation

$$z^3 + az^2 - 4b = 0;$$

then
$$y^3 + amy^2 - 4bm^3 = 0,$$

or, if
$$m^3 = \frac{1}{b} = T,$$

then
$$y^3 + amy^2 - 4 = 0;$$

and, comparing this with

$$\mathrm{p}^3 u + 3\mathrm{p}\,(iu\sqrt{3})\,\mathrm{p}^2 u - 4 = 0,$$

when
$$g_2 = 0, \quad g_3 = 4;$$

then
$$\mathrm{p}\,u = y,$$

if
$$\mathrm{p}\,(iu\sqrt{3}) = \tfrac{1}{3}am = \frac{S}{3T^{\frac{1}{3}}} = \left(\frac{J}{J-1}\right)^{\frac{1}{3}};$$

and then
$$z = \frac{\mathrm{p}\,u}{T^{\frac{1}{3}}}, \quad x = \frac{T^{\frac{1}{3}}}{\mathrm{p}\,u}.$$

Then, if two roots of the equation are imaginary, the value of $\dfrac{S}{3T^{\frac{1}{3}}}$ lies between $-\infty$ and 1; and, to solve the cubic, look out the value of $\mathrm{p}\,(iu\sqrt{3})$ corresponding to $\dfrac{S}{3T^{\frac{1}{3}}}$, and then the corresponding values of $\mathrm{p}\,u$ on the same horizontal line; and then the value of x is $\dfrac{T^{\frac{1}{3}}}{\mathrm{p}\,u}$; the other two values of x being $\dfrac{T^{\frac{1}{3}}}{\mathrm{p}\,(u \pm \frac{2}{3}\omega_2')}$.

If the three roots are real, the value of $\dfrac{S}{3T^{\frac{1}{3}}}$ lies between ∞ and 1; so that $iu\sqrt{3}$ is real; and therefore, putting

$$iu\sqrt{3} = v,$$

and looking out the value of v corresponding to

$$\mathrm{p}\,v = \frac{S}{3T^{\frac{1}{3}}},$$

the value of x will be $\dfrac{T^{\frac{1}{3}}}{\mathrm{p}\,\frac{1}{3}u}$; the other two roots being $\dfrac{T^{\frac{1}{3}}}{\mathrm{p}\,(\frac{1}{3}u \pm \frac{2}{3}\omega_2)}$.

This method of solution of the cubic has the advantage of requiring only the tabulated values of a doubly periodic function of finite periods.

B. *Solution of the Quartic Equation.*

12. Next, suppose the general quartic equation

$$U_x = (a, b, c, d, e)(x, 1)^4 = 0$$

is presented for solution.

Denoting the Hessian, changing however its sign to what is usually employed, by

$$H_x = (b^2 - ac)\,x^4 - 2\,(ad - bc)\,x^3 + (3c^2 - ae - 2bd)\,x^2$$
$$-2\,(be - cd)\,x + d^2 - ce\,;$$

then, if G_x denote the sextic covariant,

$$G_x^2 = 4H_x^3 - g_2 H_x U_x^2 - g_3 U_x^3,$$

where
$$g_2 = ae - 4bd + 3c^2,$$
$$g_3 = ace + 2bcd - ad^2 - eb^2 - c^3,$$
$$= \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix};$$

the *quadrinvariant* and the *cubinvariant* respectively.

Then, if we put
$$s = \frac{H_x}{U_x},$$
$$\frac{ds}{dx} = \frac{H'_x U_x - H_x U'_x}{U_x^2}$$
$$= 2\,\frac{G_x}{U_x^2}$$

(Cayley, *Elliptic Functions*, page 347),

and
$$4s^3 - g_2 s - g_3 = \frac{G_x^2}{U_x^3};$$

so that, if we put the general elliptic integral

$$\int \frac{dx}{\sqrt{(U_x)}} = u,$$

then
$$du = \frac{dx}{\sqrt{(U_x)}} = \tfrac{1}{2}\,\frac{ds}{\sqrt{(4s^3 - g_2 s - g_3)}};$$

or
$$2u = \int \frac{ds}{\sqrt{(4s^3 - g_2 s - g_3)}};$$

so that, in Weierstrass's notation, we may put

$$s = \frac{H_x}{U_x} = \mathrm{p}\ (2u;\ g_2, g_3);$$

and

$$-\frac{ds}{du} = 2\frac{G_x}{U_x^{\frac{3}{2}}} = 2\mathrm{p}'\ (2u;\ g_2, g_3).$$

We may, therefore, use the notation

$$u = \int \frac{dx}{\sqrt{(U)}} = \tfrac{1}{2}\mathrm{p}^{-1}\left(\frac{H}{U};\ g_2, g_3\right),$$

expressing the general elliptic integral as a function of the *covariants* U and H.

Mr. Robert Russell, of Trinity College, Dublin, has also shown how to reduce the general elliptic integral to one of Legendre's or Jacobi's canonical form, as a function of the quotients of the quadratic factors of the sextic covariant G, the squares of these quadratic factors being

$$H_x - e_1 U_x,\quad H_x - e_2 U_x,\quad H_x - e_3 U_x.$$

13. Suppose, now, that $x = \infty$ when $u = \alpha$, then

$$\mathrm{p}\,2\alpha = \frac{b^2 - ac}{a}.$$

On the assumption that it is possible to express x as a linear function of $\mathrm{p}u$, then the roots of the quartic will correspond to the infinite values of $\mathrm{p}\,2u$; so that $x = x_0, x_1, x_2, x_3$, the roots of the quartic, when $u = 0$, $\omega_1, \omega_2, \omega_3$, in the notation of Weierstrass, previously explained.

Thus, when

$$u = \int_{x_0}^{x} \frac{dx}{\sqrt{(U_x)}},$$

we have

$$\mathrm{p}2u = \frac{H_x}{U_x};$$

x_0 denoting the root of the quartic $U_x = 0$, corresponding to $u = 0$.

To express $\mathrm{p}u$ as a function of x, we can employ Klein's formula (54) (*Hyperelliptische Sigmafunctionen*, *Math. Ann.*, XXVII., p. 454),

$$\mathrm{p}u = \frac{\sqrt{\dot{U}^x}\sqrt{U_{x_0}} + \frac{1}{12}\left(x_0\frac{\delta}{\delta x} + y_0\frac{\delta}{\delta y}\right)^2 U_{(x,y)}}{2\,(x - x_0)^2},$$

where y and y_0 are replaced by unity after differentiation; and then, when x_0 is a root of $U_x = 0$,

$$\mathrm{p}u = \frac{(a, b, c)(x_0, 1)^2 x^2 + 2\,(b, c, d)(x_0, 1)^2 x + (c, d, e)(x_0, 1)^2}{2\,(x - x_0)^2};$$

and, by (55),

$$\mathfrak{p}'u = -\frac{(a,\ b,\ c,\ d)(x_0,\ 1)^3 x + (b,\ c,\ d,\ e)(x_0,\ 1)^3}{(x-x_0)^3} U_x.$$

Then, supposing $x = \infty$ when $u = \alpha$,

$$\mathfrak{p}\,\alpha = \tfrac{1}{2}(a,\ b,\ c)(x_0,\ 1)^2,$$

$$\mathfrak{p}'\alpha = -\sqrt{a}\,(a,\ b,\ c,\ d)(x_0,\ 1)^3;$$

so that
$$\mathfrak{p}u - \mathfrak{p}\,\alpha = \frac{(a,\ b,\ c,\ d)(x_0,\ 1)^3}{x-x_0}$$

$$= -\frac{\mathfrak{p}'\alpha}{\sqrt{a}\,(x-x_0)};$$

or
$$x - x_0 = -\frac{\mathfrak{p}'\alpha}{\sqrt{a}\,(\mathfrak{p}u - \mathfrak{p}\,\alpha)}.$$

Or, otherwise, putting

$$x - x_0 = \frac{A}{\mathfrak{p}u - \mathfrak{p}\alpha},$$

where A is some constant, to be determined hereafter; and then, with the notation previously explained, we can put

$$x - x_1 = \frac{A}{\mathfrak{p}u - \mathfrak{p}\alpha}\,\frac{\mathfrak{p}u - e_1}{\mathfrak{p}\alpha - e_1},$$

$$x - x_2 = \frac{A}{\mathfrak{p}u - \mathfrak{p}\alpha}\,\frac{\mathfrak{p}u - e_2}{\mathfrak{p}\alpha - e_2},$$

$$x - x_3 = \frac{A}{\mathfrak{p}u - \mathfrak{p}\alpha}\,\frac{\mathfrak{p}u - e_3}{\mathfrak{p}\alpha - e_3}.$$

Then
$$U^2 = a\,(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$= a\,\frac{A^4}{(\mathfrak{p}u - \mathfrak{p}\alpha)^4}\,\frac{\mathfrak{p}'^2 u}{\mathfrak{p}'^2\alpha}.$$

Also
$$\frac{dx}{du} = -\frac{A\mathfrak{p}'u}{(\mathfrak{p}u - \mathfrak{p}\alpha)^2};$$

so that
$$du = \frac{dx}{\sqrt{(U)}} = \frac{-\dfrac{A\mathfrak{p}'u}{(\mathfrak{p}u-\mathfrak{p}\alpha)^2}\,du}{\sqrt{a}\,\dfrac{A^2}{(\mathfrak{p}u-\mathfrak{p}\alpha)^2}\,\dfrac{\mathfrak{p}'u}{\mathfrak{p}'\alpha}},$$

and therefore
$$A = -\frac{\mathfrak{p}'\alpha}{\sqrt{a}}.$$

Then
$$x_0-x_1=\frac{A}{\mathrm{p}a-e_1}=-\frac{1}{\sqrt{a}}\,\frac{\mathrm{p}'a}{\mathrm{p}a-e_1},$$
$$x_0-x_2=\frac{A}{\mathrm{p}a-e_2}=-\frac{1}{\sqrt{a}}\,\frac{\mathrm{p}'a}{\mathrm{p}a-e_2},$$
$$x_0-x_3=\frac{A}{\mathrm{p}a-e_3}=-\frac{1}{\sqrt{a}}\,\frac{\mathrm{p}'a}{\mathrm{p}a-e_3};$$

so that, since
$$x_0+x_1+x_2+x_3=-4\frac{b}{a},$$
$$3x_0-x_1-x_2-x_3=4\left(x_0+\frac{b}{a}\right)$$
$$=-\frac{1}{\sqrt{a}}\left(\frac{\mathrm{p}'a}{\mathrm{p}a-e_1}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_2}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_3}\right);$$

or
$$x_0+\frac{b}{a}=-\frac{1}{4\sqrt{a}}\left(\frac{\mathrm{p}'a}{\mathrm{p}a-e_1}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_2}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_3}\right);$$

and, therefore,
$$x_1+\frac{b}{a}=-\frac{1}{4\sqrt{a}}\left(\frac{-3\mathrm{p}'a}{\mathrm{p}a-e_1}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_2}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_3}\right),$$
$$x_2+\frac{b}{a}=-\frac{1}{4\sqrt{a}}\left(\frac{\mathrm{p}'a}{\mathrm{p}a-e_1}+\frac{-3\mathrm{p}'a}{\mathrm{p}a-e_2}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_3}\right),$$
$$x_3+\frac{b}{a}=-\frac{1}{4\sqrt{a}}\left(\frac{\mathrm{p}'a}{\mathrm{p}a-e_1}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_2}+\frac{-3\mathrm{p}'a}{\mathrm{p}a-e_3}\right).$$

Then
$$U=\frac{1}{a}\,\frac{\mathrm{p}'^2a\,\mathrm{p}'^2u}{(\mathrm{p}u-\mathrm{p}a)^4}=\frac{1}{a}\{\mathrm{p}(u-a)-\mathrm{p}(u+a)\}^2,$$

as in M. Halphen's paper "Sur l'inversion des Integrales Elliptiques," *Journal de l'Ecole Polytechnique*, 1884.

14. These preceding investigations indicate the advantage of the substitution of § 61 of Burnside and Panton's *Theory of Equations*, where the second term of the quartic
$$ax^4+4bx^3+6cx^2+4dx+e=0$$
is removed by means of the substitution
$$z=ax+b,$$
or
$$x=\frac{z}{a}-\frac{b}{a};$$

so that the quartic becomes

$$V = z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0,$$

where
$$H = ac - b^2,$$
$$G = a^2d - 3abc + 2b^3.$$

Then the quadrinvariant and the cubinvariant of V are

$$G_2 = a^2 g_2,$$
$$G_3 = a^3 g_3;$$

also
$$dv = \frac{dz}{\sqrt{(V)}} = \frac{dx}{\sqrt{(aU)}} = \frac{du}{\sqrt{a}},$$

so that
$$u = v\sqrt{a},$$

agreeing with the formula

$$\wp\,(u;\ g_2, g_3) = \frac{1}{a}\,\wp\,(v;\ G_2, G_3).$$

15. It will simplify matters, without any restriction on generality, to suppose hereafter a is replaced by unity whenever necessary, so that u and v are the same; also $G_2 = g_2$, $G_3 = g_3$; and then, α denoting the value of u which makes x, and therefore z, infinite,

$$\wp\,2\alpha = H, \quad \text{and} \quad \wp'\,2\alpha = -G;$$

and, denoting the roots of the quartic in z by z_0, z_1, z_2, z_3; then, as before,

$$z - z_0 = \frac{-\wp'\alpha}{\wp u - \wp\alpha},$$

$$z - z_1 = \frac{-\wp'\alpha}{\wp u - \wp\alpha}\,\frac{\wp u - e_1}{\wp\alpha - e_1},$$

$$z - z_2 = \frac{-\wp'\alpha}{\wp u - \wp\alpha}\,\frac{\wp u - e_2}{\wp\alpha - e_2},$$

$$z - z_3 = \frac{-\wp'\alpha}{\wp u - \wp\alpha}\,\frac{\wp u - e_3}{\wp\alpha - e_3};$$

also
$$z_0 = -\tfrac{1}{4}\left(\frac{\wp'\alpha}{\wp\alpha - e_1} + \frac{\wp'\alpha}{\wp\alpha - e_2} + \frac{\wp'\alpha}{\wp\alpha - e_3}\right),$$

$$z_1 = -\tfrac{1}{4}\left(\frac{-3\wp'\alpha}{\wp\alpha - e_1} + \frac{\wp'\alpha}{\wp\alpha - e_2} + \frac{\wp'\alpha}{\wp\alpha - e_3}\right),$$

$$z_2 = -\tfrac{1}{4}\left(\frac{\mathrm{p}'a}{\mathrm{p}a-e_1}+\frac{-3\mathrm{p}'a}{\mathrm{p}a-e_2}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_3}\right),$$

$$z_3 = -\tfrac{1}{4}\left(\frac{\mathrm{p}'a}{\mathrm{p}a-e_1}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_2}+\frac{-3\mathrm{p}'a}{\mathrm{p}a-e_3}\right).$$

16. By means of the notation explained by Schwarz in *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen*, we can transform the above expressions for the roots z_0, z_1, z_2, z_3, into

$$z_0 = 2\frac{\sigma'a}{\sigma a}-\frac{\sigma'2a}{\sigma 2a} = -\tfrac{1}{2}\frac{\mathrm{p}''a}{\mathrm{p}'a},$$

$$z_1 = 2\frac{\sigma_1'a}{\sigma_1 a}-\frac{\sigma'2a}{\sigma 2a} = -\tfrac{1}{2}\frac{\mathrm{p}''a}{\mathrm{p}'a}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_1} = -\tfrac{1}{2}\frac{\mathrm{p}''(a+\omega_1)}{\mathrm{p}'(a+\omega_1)},$$

$$z_2 = 2\frac{\sigma_2'a}{\sigma_2 a}-\frac{\sigma'2a}{\sigma 2a} = -\tfrac{1}{2}\frac{\mathrm{p}''a}{\mathrm{p}'a}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_2} = -\tfrac{1}{2}\frac{\mathrm{p}''(a+\omega_2)}{\mathrm{p}'(a+\omega_2)},$$

$$z_3 = 2\frac{\sigma_3'a}{\sigma_3 a}-\frac{\sigma'2a}{\sigma 2a} = -\tfrac{1}{2}\frac{\mathrm{p}''a}{\mathrm{p}'a}+\frac{\mathrm{p}'a}{\mathrm{p}a-e_3} = -\tfrac{1}{2}\frac{\mathrm{p}''(a+\omega_3)}{\mathrm{p}'(a+\omega_3)};$$

or, in another form,

$$z_0 = \frac{\sigma_1 2a}{\sigma 2a}+\frac{\sigma_2 2a}{\sigma 2a}+\frac{\sigma_3 2a}{\sigma 2a},$$

$$z_1 = \frac{\sigma_1 2a}{\sigma 2a}-\frac{\sigma_2 2a}{\sigma 2a}-\frac{\sigma_3 2a}{\sigma 2a},$$

$$z_2 = -\frac{\sigma_1 2a}{\sigma 2a}+\frac{\sigma_2 2a}{\sigma 2a}-\frac{\sigma_3 2a}{\sigma 2a},$$

$$z_3 = -\frac{\sigma_1 2a}{\sigma 2a}-\frac{\sigma_2 2a}{\sigma 2a}+\frac{\sigma_3 2a}{\sigma 2a};$$

equivalent to

$$z_0 = \sqrt{(\mathrm{p}2a-e_1)}+\sqrt{(\mathrm{p}2a-e_2)}+\sqrt{(\mathrm{p}2a-e_3)},$$
$$z_1 = \sqrt{(\mathrm{p}2a-e_1)}-\sqrt{(\mathrm{p}2a-e_2)}-\sqrt{(\mathrm{p}2a-e_3)},$$
$$z_2 = -\sqrt{(\mathrm{p}2a-e_1)}+\sqrt{(\mathrm{p}2a-e_2)}-\sqrt{(\mathrm{p}2a-e_3)},$$
$$z_3 = -\sqrt{(\mathrm{p}2a-e_1)}-\sqrt{(\mathrm{p}2a-e_2)}+\sqrt{(\mathrm{p}2a-e_3)},$$

agreeing with the expressions on page 117 of Burnside and Panton's *Theory of Equations*.

Then $$(x_0-x_1)(x_2-x_3) = (z_0-z_1)(z_2-z_3)$$
$$= 4(\mathrm{p}2a-e_3-\mathrm{p}2a+e_2) = 4(e_2-e_3);$$

and, similarly,

$$(x_0-x_2)(x_3-x_1) = (z_0-z_2)(z_3-z_1) = 4\,(e_3-e_1),$$

$$(x_0-x_3)(x_1-x_2) = (z_0-z_3)(z_1-z_2) = 4\,(e_1-e_2)\,;$$

e_1, e_2, e_3 denoting the roots of the *reducing cubic*

$$4s^3-g_2s-g_3=0,$$

and replacing a by unity.

17. The simplest expression, however, of the roots of a quartic is obtained by increasing the roots of the equation in z by $\dfrac{\sigma' 2a}{\sigma 2a}$, equivalent to putting

$$\frac{\sigma' 2a}{\sigma 2a} = -b$$

in the quartic equation

$$y^4+4by^3+6cy^2+4dy+e=0\,;$$

when the roots of the quartic are

$$y_0 = 2\frac{\sigma' a}{\sigma a},\quad y_1 = 2\frac{\sigma_1' a}{\sigma_1 a},\quad y_2 = 2\frac{\sigma_2' a}{\sigma_2 a},\quad y_3 = 2\frac{\sigma_3' a}{\sigma_3 a}.$$

Then $\quad y_1-y_0 = \dfrac{\wp' a}{\wp a-e_1},\quad y_2-y_0 = \dfrac{\wp' a}{\wp a-e_2},\quad y_3-y_0 = \dfrac{\wp' a}{\wp a-e_3}\,;$

so that $\quad (y_1-y_0)(y_2-y_3) = \dfrac{\wp'^2 a\,(e_2-e_3)}{(\wp a-e_1)(\wp a-e_2)(\wp a-e_3)} = 4\,(e_2-e_3),$

and, similarly, $\quad (y_2-y_0)(y_3-y_1) = 4\,(e_3-e_1),$

$$(y_3-y_0)(y_1-y_2) = 4\,(e_1-e_2).$$

Then, generally, $\quad y = \dfrac{\sigma'(u+a)}{\sigma(u+a)} - \dfrac{\sigma'(u-a)}{\sigma(u-a)},$

if $\quad u = \displaystyle\int \frac{dy}{\sqrt{(U_y)}}.$

18. We may now compare these substitutions with those given by Halphen (*Journal de l'Ecole Polytechnique*, 1884), where

$$z = \tfrac{1}{2}\,\frac{\wp' u-\wp' v}{\wp u-\wp v}$$

$$= \frac{\sigma'(u+v)}{\sigma(u+v)} - \frac{\sigma' u}{\sigma u} - \frac{\sigma' v}{\sigma v}\,;$$

or, in our notation, with

$$z = \tfrac{1}{2}\frac{\wp'(u-a)-\wp' 2a}{\wp(u-a)-\wp 2a}$$

$$= \tfrac{1}{2}\frac{\wp'(u+a)+\wp' 2a}{\wp(u+a)-\wp 2a} = \tfrac{1}{2}\frac{\wp'(u-a)-\wp'(u+a)}{\wp(u-a)-\wp(u+a)}$$

$$= \frac{\sigma'(u+a)}{\sigma(u+a)} - \frac{\sigma'(u-a)}{\sigma(u-a)} - \frac{\sigma' 2a}{\sigma 2a},$$

or
$$y = \frac{\sigma'(u+a)}{\sigma(u+a)} - \frac{\sigma'(u-a)}{\sigma(u-a)},$$

with the notation of the last article (§ 16).

Then
$$V = \frac{\wp'^2 a\, \wp'^2 u}{(\wp u - \wp a)^4}$$

$$= \{\wp(u-a)-\wp(u+a)\}^2,$$

and
$$U = \frac{1}{a}\{\wp(u-a)-\wp(u+a)\}^2,$$

or
$$\sqrt{U} = \frac{1}{\sqrt{a}}\{\wp(u-a)-\wp(u+a)\},$$

agreeing with Halphen's expression.

19. It remains to investigate the conditions for the reality or otherwise of the roots of the quartic.

(A) When the discriminant

$$\Delta = g_2^3 - 27 g_3^2$$

is negative, two of the roots e_1, e_2, e_3 of the reducing cubic

$$4s^3 - g_2^3 - g_3 = 0$$

are imaginary; and then two of the roots of the quartic are imaginary and two are real.

(B) When the discriminant Δ is positive, e_1, e_2, e_3 are all real, and the roots of the quartic may be all real, or all imaginary.

The roots will be all real when $\wp c$ and $\wp' c$ are real; that is, when $\wp 2c$ or H lies between ∞ and e_1; otherwise, all the roots will be imaginary.

The solution of the *quintic* by means of Weierstrass's functions has been considered by Kiepert in *Crelle*, Vol. 87, page 120.

20. Let us apply the preceding theories to the motion of a prolate solid of revolution, moving through infinite liquid, under no forces.

Then, as explained in the *Quar. Jour. of Math.* (No. 62, 1879), if $\cos\theta$ is denoted by x,

$$\dot{x}^2 = a_0(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$
$$= a_0x^4 - 4a_1x^3 + 6a_2x^2 - 4a_3x + a_4,$$

where $$a_0 = \frac{F^2}{c_4}\left(\frac{1}{c_3}-\frac{1}{c_1}\right) = M, \quad \text{also } a_1 = 0;$$

and if $$\wp 2c = \frac{a_2}{a_0}, \quad \wp' 2c = \frac{a_3}{a_0};$$

then $$x_0 = -\tfrac{1}{2}\frac{\wp'' c}{\wp' c}, \quad x_1 = -\tfrac{1}{2}\frac{\wp''(c+\omega_1)}{\wp'(c+\omega_1)}, \quad x_2 = \ldots, \quad x_3 = \ldots;$$

and $$x = \frac{\sigma'(u+c)}{\sigma(u+c)} - \frac{\sigma'(u-c)}{\sigma(u-c)} - \frac{\sigma' 2c}{\sigma 2c}$$
$$= \tfrac{1}{2}\frac{\wp'(u-c) - \wp' 2c}{\wp(u-c) - \wp 2c},$$

and $$x - x_0 = \frac{\sigma'(u+c)}{\sigma(u+c)} - \frac{\sigma'(u-c)}{\sigma(u-c)} - 2\frac{\sigma' c}{\sigma c}$$
$$= \frac{-\wp' c}{\wp u - \wp c},$$

$$x - x_1 = \frac{-\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_1}{\wp c - e_1},$$

$$x - x_2 = \frac{-\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_2}{\wp c - e_2},$$

$$x - x_3 = \frac{-\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_3}{\wp c - e_3};$$

so that $$(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$
$$= \frac{\wp'^4 c}{(\wp u - \wp c)^4}\,\frac{\wp'^2 u}{\wp'^2 c}$$
$$= \frac{\wp'^2 c\, \wp'^2 u}{(\wp u - \wp c)^4}$$
$$= \{\wp(u-c) - \wp(u+c)\}^2;$$

also $$\frac{dx}{du} = \frac{\wp' c\, \wp' u}{(\wp u - \wp c)^2} = \wp(u-c) - \wp(u+c),$$

and therefore $$du = \sqrt{a_0}\,dt,$$

or $$u = \sqrt{a_0}\,t + \text{constant}.$$

The constant must be taken to be ω_3, for $\wp u$ to range between e_2 and e_3, and therefore x to range between x_2 and x_3.

21. Then $$\frac{d\psi}{dt} = \frac{G+c_6 n}{2c_4}\,\frac{1}{1+\cos\theta} + \frac{G-c_6 n}{2c_3}\,\frac{1}{1-\cos\theta},$$

$$\frac{d\psi}{du} = \tfrac{1}{2}\,\frac{G+c_6 n}{2c_4\sqrt{M}}\,\frac{1}{1+\cos\theta} + \frac{G-c_6 n}{2c_4\sqrt{M}}\,\frac{1}{1-\cos\theta},$$

$$= \tfrac{1}{2}i\,\frac{\sqrt{(1+x_0\,.\,1+x_1\,.\,1+x_2\,.\,1+x_3)}}{1+\cos\theta} + \tfrac{1}{2}i\,\frac{\sqrt{(1-x_0\,.\,1-x_1\,.\,1-x_2\,.\,1-x_3)}}{1-\cos\theta}.$$

Now, suppose $$u = a \quad \text{when} \quad \cos\theta = -1,$$
$$u = b \quad \text{,,} \quad \cos\theta = +1;$$

then $$1+\cos\theta = \frac{\wp' c}{\wp a - \wp c} - \frac{\wp' c}{\wp u - \wp c} = \frac{\wp' c\,(\wp u - \wp a)}{(\wp a - \wp c)(\wp u - \wp c)},$$

$$1-\cos\theta = \frac{\wp' c}{\wp u - \wp c} - \frac{\wp' c}{\wp b - \wp c} = \frac{\wp' c\,(\wp b - \wp u)}{(\wp b - \wp c)(\wp u - \wp c)},$$

$$1+x_0\,.\,1+x_1\,.\,1+x_2\,.\,1+x_3 = \frac{\wp'^2 a\,\wp'^2 c}{(\wp a - \wp c)^4},$$

$$1-x_0\,.\,1-x_1\,.\,1-x_2\,.\,1-x_3 = \frac{\wp'^2 b\,\wp'^2 c}{(\wp b - \wp c)^4};$$

also $\wp' a$ is negative imaginary, and $\wp' b$ positive imaginary,

so that $$\frac{d\psi}{du} = \tfrac{1}{2}i\,\frac{\dfrac{\wp' a\,\wp' c}{(\wp a - \wp c)^2}}{\dfrac{\wp' c\,(\wp u - \wp a)}{(\wp a - \wp c)(\wp u - \wp c)}} - \tfrac{1}{2}i\,\frac{\dfrac{\wp' b\,\wp' c}{(\wp b - \wp c)^2}}{\dfrac{\wp' c\,(\wp b - \wp u)}{(\wp b - \wp c)(\wp u - \wp c)}}$$

$$= \tfrac{1}{2}i\,\frac{\wp' a\,(\wp u - \wp c)}{(\wp a - \wp c)(\wp u - \wp a)} + \tfrac{1}{2}i\,\frac{\wp' b\,(\wp u - \wp c)}{(\wp b - \wp c)(\wp u - \wp b)},$$

$$2\frac{d\psi}{diu} = \frac{\wp' a}{\wp a - \wp c} + \frac{\wp' a}{\wp u - \wp a} - \frac{\wp' b}{\wp b - \wp c} - \frac{\wp' b}{\wp u - \wp b}$$

$$= \frac{\sigma'}{\sigma}(a+c) + \frac{\sigma'}{\sigma}(a-c) - 2\frac{\sigma'}{\sigma}a - \frac{\sigma'}{\sigma}(u+a) + \frac{\sigma'}{\sigma}(u-a) + 2\frac{\sigma'}{\sigma}a,$$

$$+\frac{\sigma'}{\sigma}(b+c) - \frac{\sigma'}{\sigma}(b-c) + 2\frac{\sigma'}{\sigma}b - \frac{\sigma'}{\sigma}(u+b) + \frac{\sigma'}{\sigma}(u-b) - 2\frac{\sigma'}{\sigma}b;$$

so that $$\psi = \tfrac{1}{2}i \log \frac{\sigma(u-a)\,\sigma(u-b)}{\sigma(u+a)\,\sigma(u+b)} + \tfrac{1}{2}iPu,$$

where $$P = \frac{\sigma'}{\sigma}(a+c) + \frac{\sigma'}{\sigma}(a-c) + \frac{\sigma'}{\sigma}(b+c) + \frac{\sigma'}{\sigma}(b-c),$$

or $$e^{-2i\psi} = e^{Pu} \frac{\sigma(u-a)\,\sigma(u-b)}{\sigma(u+a)\,\sigma(u+b)}.$$

22. Then

$$x_0 = -\tfrac{1}{2}\frac{\wp'' c}{\wp' c} = 2\frac{\sigma_0'}{\sigma_0}c - \frac{\sigma'}{\sigma}2c = \frac{\sigma_1}{\sigma}2c + \frac{\sigma_2}{\sigma}2c + \frac{\sigma_3}{\sigma}2c,$$

$$x_1 = -\tfrac{1}{2}\frac{\wp''(c+\omega_1)}{\wp'(c+\omega_1)} = 2\frac{\sigma_1'}{\sigma_1}c - \frac{\sigma'}{\sigma}2c = \frac{\sigma_1}{\sigma}2c - \frac{\sigma_2}{\sigma}2c - \frac{\sigma_3}{\sigma}2c,$$

$$x_2 = -\tfrac{1}{2}\frac{\wp''(c+\omega_2)}{\wp'(c+\omega_2)} = 2\frac{\sigma_2'}{\sigma_2}c - \frac{\sigma'}{\sigma}2c = -\frac{\sigma_1}{\sigma}2c + \frac{\sigma_2}{\sigma}2c - \frac{\sigma_3}{\sigma}2c,$$

$$x_3 = -\tfrac{1}{2}\frac{\wp''(c+\omega_3)}{\wp'(c+\omega_3)} = 2\frac{\sigma_3'}{\sigma_3}c - \frac{\sigma'}{\sigma}2c = -\frac{\sigma_1}{\sigma}2c - \frac{\sigma_2}{\sigma}2c + \frac{\sigma_3}{\sigma}2c;$$

and, generally,

$$x = \tfrac{1}{2}\frac{\wp'(u-c) - \wp' 2c}{\wp(u-c) - \wp\, 2c} = \frac{\sigma'}{\sigma}(u+c) - \frac{\sigma'}{\sigma}(u-c) - \frac{\sigma'}{\sigma}2c,$$

$$\frac{dx}{dt} = \sqrt{M}\frac{\wp' c\, \wp' u}{(\wp u - \wp c)^2} = \sqrt{M}\,\{\wp(u-c) - \wp(u+c)\}.$$

23. In order to agree with the notation of the *Quarterly Journal*, we must suppose

$$x_0 = \delta, \quad x_1 = \alpha, \quad x_2 = \beta, \quad x_3 = \gamma;$$

and then $$(\alpha-\gamma)(\beta-\delta) = 4(e_1 - e_3);$$

also, in order for $\wp u$ to oscillate in value between e_2 and e_3, we must have $$u = \sqrt{M}\,t + \omega_3.$$

Comparing Weierstrass's notation with Jacobi's, we have

$$\mathrm{p}u-e_1=\left(\frac{\sigma_1 u}{\sigma u}\right)^2=(e_1-e_3)\,\frac{\mathrm{cn}^2}{\mathrm{dn}^2}\sqrt{(e_1-e_3)}\,u$$

or

$$(e_1-e_3)\,\mathrm{cd}^2\sqrt{(e_1-e_3)}\,u;$$

$$\mathrm{p}u-e_2=\left(\frac{\sigma_2 u}{\sigma u}\right)^2=(e_1-e_3)\,\mathrm{ds}^2\sqrt{(e_1-e_3)}\,u,$$

$$\mathrm{p}u-e_3=\left(\frac{\sigma_3 u}{\sigma u}\right)^2=(e_1-e_3)\,\mathrm{ns}^2\sqrt{(e_1-e_3)}\,u$$

(Schwarz, page 30);

and $$\sqrt{(e_1-e_3)}\,u=\tfrac{1}{2}\sqrt{\{M(\alpha-\gamma)(\beta-\delta)\}}\,t+2iK',$$

agreeing with the notation of the *Quarterly Journal.*

The determination of α, β, γ, δ attempted in that article has thus been effected, in terms of Weierstrass functions of c, the invariants being g_2 and g_3, the invariants of the quartic $\dot{x}^2$ in terms of x.

24. Now $$1+\delta=\frac{\mathrm{p}'c}{\mathrm{p}a-\mathrm{p}c},\quad 1-\delta=\frac{-\mathrm{p}'c}{\mathrm{p}b-\mathrm{p}c};$$

so that $$\frac{1-\delta}{1+\delta}=-\frac{\mathrm{p}a-\mathrm{p}c}{\mathrm{p}b-\mathrm{p}c};$$

and, similarly, $$\frac{1-\alpha}{1+\alpha}=-\frac{\mathrm{p}a-\mathrm{p}c}{\mathrm{p}b-\mathrm{p}c}\,\frac{\mathrm{p}b-e_1}{\mathrm{p}a-e_1},$$

$$\frac{1-\beta}{1+\beta}=-\frac{\mathrm{p}a-\mathrm{p}c}{\mathrm{p}b-\mathrm{p}c}\,\frac{\mathrm{p}b-e_2}{\mathrm{p}a-e_2},$$

$$\frac{1-\gamma}{1+\gamma}=-\frac{\mathrm{p}a-\mathrm{p}c}{\mathrm{p}b-\mathrm{p}c}\,\frac{\mathrm{p}b-e_3}{\mathrm{p}a-e_3};$$

so that $$\frac{\dfrac{1-\alpha}{1+\alpha}}{\dfrac{\sigma_1^2 b}{\sigma_1^2 a}}=\frac{\dfrac{1-\beta}{1+\beta}}{\dfrac{\sigma_2^2 b}{\sigma_2^2 a}}=\frac{\dfrac{1-\gamma}{1+\gamma}}{\dfrac{\sigma_3^2 b}{\sigma_3^2 a}}=\frac{\dfrac{1-\delta}{1+\delta}}{\dfrac{\sigma^2 b}{\sigma^2 a}},$$

and if this is put $=\dfrac{1}{x}$, and

$$A=\frac{\sigma_1^2 b}{\sigma_1^2 a},\quad B=\frac{\sigma_2^2 b}{\sigma_2^2 a},\quad C=\frac{\sigma_3^2 b}{\sigma_3^2 a},\quad D=\frac{\sigma^2 b}{\sigma^2 a},$$

the biquadratic

$$4x^4-2x^3(A+B+C+D)$$
$$+2x(BCD+CDA+DAB+ABC)-4ABCD=0,$$

obtained by putting $\alpha+\beta+\gamma+\delta=0$,

has a root $$x_0 = -\frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma^2 b}{\sigma^2 a} = -\frac{\sigma(a+c)\,\sigma(a-c)}{\sigma(b+c)\,\sigma(b-c)},$$

the other roots being

$$-\frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma_1^2 b}{\sigma_1^2 a},\quad -\frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma_2^2 b}{\sigma_2^2 a},\quad -\frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma_3^2 b}{\sigma_3^2 a}.$$

Denoting these four roots by x_0, x_1, x_2, x_3; then

$$x_1-x_0 = \frac{(\wp a-\wp c)}{(\wp b-\wp c)}\frac{\sigma^2 b}{\sigma^2 a}\frac{\wp a-\wp b}{\wp a-e_1},$$

$$x_2-x_3 = \frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma^2 b}{\sigma^2 a}\frac{(\wp a-\wp b)(e_2-e_3)}{(\wp a-e_2)(\wp a-e_3)};$$

so that

$$(x_1-x_0)(x_2-x_3) = \left(\frac{\wp a-\wp c}{\wp b-\wp c}\right)^2\frac{\sigma^4 b}{\sigma^4 a}\frac{(\wp a-\wp b)^2}{\wp'^2 a}4\,(e_2-e_3),$$

so that the S and T of the reducing cubic of the last quartic becomes

$$S = m^4 g_2,\quad T = m^4 g_3,$$

where $$m = \frac{\wp a-\wp c}{\wp b-\wp c}\frac{\sigma^2 b}{\sigma^2 a}\frac{\wp a-\wp b}{\wp' a};$$

and this complication is sufficient to explain the difficulty experienced previously in the attempt to solve the biquadratic (6) and its reducing cubic.

25. Compared with the previous expressions in the *Quarterly Journal*, Vol. XVI.,

$$\operatorname{sn}^2 ia' = \frac{\alpha-\gamma}{\alpha-\delta}\frac{1+\delta}{1+\gamma} = \frac{\dfrac{1+\alpha}{1+\gamma}-1}{\dfrac{1+\alpha}{1+\delta}-1} = \frac{\dfrac{\wp a-e_1}{\wp c-e_1}\dfrac{\wp c-e_3}{\wp a-e_3}-1}{\dfrac{\wp a-e_1}{\wp c-e_1}-1} = \frac{e_1-e_3}{\wp a-e_3},$$

$$\operatorname{cn}^2 ia' = \frac{\gamma-\delta}{\alpha-\delta}\frac{1+\alpha}{1+\gamma} = \frac{1-\dfrac{1+\delta}{1+\gamma}}{1-\dfrac{1+\delta}{1+\alpha}} = \frac{1-\dfrac{\wp c-e_3}{\wp a-e_3}}{1-\dfrac{\wp c-e_1}{\wp a-e_1}} = \frac{\wp a-e_1}{\wp a-e_3},$$

$$\operatorname{dn}^2 ia' = \frac{\gamma-\delta}{\beta-\delta}\frac{1+\beta}{1+\gamma} = \frac{1-\dfrac{1+\delta}{1+\gamma}}{1-\dfrac{1+\delta}{1+\beta}} = \frac{1-\dfrac{\wp c-e_3}{\wp a-e_3}}{1-\dfrac{\wp c-e_2}{\wp a-e_2}} = \frac{\wp a-e_2}{\wp a-e_3},$$

and similarly
$$\mathrm{sn}^2(ib'+K) = \frac{e_1-e_3}{\mathrm{p}b-e_3},$$
$$\mathrm{cn}^2(ib'+K) = \frac{\mathrm{p}a-e_1}{\mathrm{p}a-e_3},$$
$$\mathrm{dn}^2(ib'+K) = \frac{\mathrm{p}a-e_2}{\mathrm{p}a-e_3};$$

indicating that
$$a = r\omega_3, \quad b = \omega_1+s\omega_3,$$

where r and s are proper fractions, with Schwarz's notation (*Formeln*, p. 74). Also, to the complementary modulus,

$$\mathrm{sn}^2 a' = \frac{1-\dfrac{1+\gamma}{1+\alpha}}{1-\dfrac{1+\gamma}{1+\delta}} = \frac{1-\dfrac{\mathrm{p}a-e_3}{\mathrm{p}c-e_3}\dfrac{\mathrm{p}c-e_1}{\mathrm{p}a-e_1}}{1-\dfrac{\mathrm{p}a-e_3}{\mathrm{p}c-e_3}} = -\frac{e_1-e_3}{\mathrm{p}a-e_1}$$

$$\mathrm{cn}^2 a' = \frac{\mathrm{p}a-e_3}{\mathrm{p}a-e_1},$$

$$\mathrm{dn}^2 a' = \frac{\mathrm{p}a-e_2}{\mathrm{p}a-e_1},$$

$$\mathrm{sn}^2 b' = \frac{1-\dfrac{1-\beta}{1-\delta}}{1-\dfrac{1-\beta}{1-\alpha}} = \frac{1-\dfrac{\mathrm{p}b-e_2}{\mathrm{p}c-e_2}}{1-\dfrac{\mathrm{p}b-e_2}{\mathrm{p}c-e_2}\dfrac{\mathrm{p}c-e_1}{\mathrm{p}b-e_1}} = -\frac{\mathrm{p}b-e_1}{e_1-e_2},$$

$$\mathrm{cn}^2 b' = \frac{\mathrm{p}b-e_2}{e_1-e_2},$$

$$\mathrm{dn}^2 b' = \frac{\mathrm{p}b-e_3}{e_1-e_3}.$$

26. Now put $x = \cos\theta = \dfrac{1-y}{1+y}$, so that $y = \tan^2\tfrac{1}{2}\theta$;

then
$$4\dot{y}^2 = -(A_0y^4-4A_1y^3+6A_2y^2-4A_3y+A_4);$$

or, if $x = \dfrac{z-1}{z+1}$, so that $z = \cot^2\tfrac{1}{2}\theta$,

then
$$4\dot{z}^2 = -(A_4z^4-4A_3z^3+6A_2z^2-4A_1z+A_0),$$

where
$$A_0 = \frac{(G+c_6n)^2}{c_4^2}, \quad A_4 = \frac{(G-c_6n)^2}{c_4^2}.$$

Then the quadrinvariant G_2 and the cubinvariant G_3 of these reciprocal quartics in y and z are the same, and

$$G_2 = 2^4 g_2, \quad G_3 = -2^6 g_3;$$

so that, if $y = \infty$ when $t = a$, $z = \infty$ when $t = b$, we may put, changing from G_2 and G_3 to g_2 and g_3,

$$\sqrt{A_0}\, y = \frac{\sigma'}{\sigma} \tfrac{1}{2}(t+a) - \frac{\sigma'}{\sigma} \tfrac{1}{2}(t-a),$$

$$\sqrt{A_4}\, z = \frac{\sigma'}{\sigma} \tfrac{1}{2}(t+b) - \frac{\sigma'}{\sigma} \tfrac{1}{2}(t-b),$$

and $$\frac{d\psi}{dt} = \tfrac{1}{4}\sqrt{A_0}\sec^2 \tfrac{1}{2}\theta + \tfrac{1}{4}\sqrt{A_4}\operatorname{cosec}^2 \tfrac{1}{2}\theta$$

$$= \tfrac{1}{4}(A_0+A_4) + \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t+a) - \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t-a)$$

$$+ \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t+b) - \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t-b),$$

or $$\frac{d\psi}{dt} = \tfrac{1}{2}\frac{G}{c_4} + \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t+a) - \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t-a)$$

$$+ \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t+b) - \tfrac{1}{4}\frac{\sigma'}{\sigma}\tfrac{1}{2}(t-b),$$

so that $$\psi = \tfrac{1}{2}\frac{G}{c_4}t + \tfrac{1}{2}\log\frac{\sigma\,\tfrac{1}{2}(t+a)}{\sigma\,\tfrac{1}{2}(t-a)} + \tfrac{1}{2}\log\frac{\sigma\,\tfrac{1}{2}(t+b)}{\sigma\,\tfrac{1}{2}(t-b)}.$$

Also $$\sqrt{A_0}\, y_0 = 2\frac{\sigma'}{\sigma}\tfrac{1}{2}a, \quad \sqrt{A_4}\, z_0 = 2\frac{\sigma'}{\sigma}\tfrac{1}{2}b;$$

so that $$\sqrt{A_0}\,(y-y_0) = \frac{-\mathrm{p}'a}{\mathrm{p}u - \mathrm{p}a},$$

with G_2 and G_3.

Arranged in descending order of magnitude, we have

$$\infty > x_1 > 1 > x_2 > x > x_3 > -1 > x_0 > -\infty;$$

also, when
$$\begin{array}{llll} x = \infty, & y = -1, & z = -1, & u = c; \\ x = x_1, & y = y_1, & z = z_1, & u = \omega_1; \\ x = 1, & y = 0, & z = \infty, & u = b; \\ x = x_2, & y = y_2, & z = z_2, & u = \omega_2; \end{array}$$

and now x, y, z have the real values of the problem; also

$$\begin{array}{llll} x = x_3, & y = y_3, & z = z_3, & u = \omega_3; \\ x = -1, & y = \infty, & z = 0, & u = a; \\ x = x_0, & y = y_0, & z = z_0, & u = 0; \\ x = -\infty, & y = -1, & z = -1, & u = c. \end{array}$$

Then
$$\dot{x}^2 = M(x-x_0)(x-x_1)(x-x_2)(x-x_3),$$
$$4\dot{y}^2 = -A_0(y-y_0)(y-y_1)(y-y_2)(y-y_3),$$
$$4\dot{z}^2 = -A_4(z-z_0)(z-z_1)(z-z_2)(z-z_3),$$

and now $\wp a$ lies between e_1 and e_2 and $\wp' a$ is positive imaginary,

$\wp b$ „ e_3 and $-\infty$ and $\wp' b$ is negative „

$\wp u$ „ e_2 and e_3,

and
$$u = t + \omega_3.$$

Appendix.

Let us apply the Table to the solution of two representative Cubic Equations.

(i.) To solve $x^3+x^2-1=0$, a modular equation of the 23rd order, the real root of which is $\sqrt[12]{(16kk')}$, when $K'/K = \sqrt{23}$.

Here
$$a = 1, \quad b = \tfrac{1}{4}, \quad m = \sqrt[12]{4};$$
$$\wp(iu\sqrt{3}) = \tfrac{1}{3}am = \cdot 529;$$

so that, to the nearest integral value of r,
$$iu\sqrt{3} = \tfrac{143}{180}\omega_2', \quad u = \tfrac{143}{180}\omega_2,$$
$$\wp u = \cdot 07882, \quad \text{and} \quad z = \cdot 7551.$$

(ii.) To solve $x^3-5x^2+6x-1=0$, the roots of which are
$$4\sin^2\frac{\pi}{14}, \quad 4\sin^2\frac{3\pi}{14}, \quad 4\sin^2\frac{5\pi}{14}.$$

Put $x = \dfrac{z}{2z+1}$, then
$$7z^3+7z^2-1=0;$$
here
$$a = 1, \quad m = \sqrt[3]{28};$$

and $$\wp(iu\sqrt{3}) = \tfrac{1}{3}\sqrt[3]{28} = 1{\cdot}012,$$

so that, to the nearest integral value of r,

$$iu\sqrt{3} = \tfrac{171}{180}\omega_2\,(\pm 2\omega_2), \quad u = \tfrac{57}{180}\omega_2'\,(\pm \tfrac{2}{3}\omega_2');$$

$$\wp u = -2{\cdot}226, \quad +{\cdot}996, \quad -1{\cdot}803;$$

and $$x = 1{\cdot}573, \quad {\cdot}198, \quad 3{\cdot}16.$$

On the Cremonian Congruences which are contained in a Linear Complex. By Dr. T. Archer Hirst, F.R.S.

[*Read May 13th*, 1886.]

1. In his well-known memoir,* published in the *Monats Bericht* of the Academy of Berlin (17th January, 1878), Kummer drew attention to the existence of two different, and equally general, congruences of the third order and third class. One of these is contained in a linear complex; the other, which for distinction might be termed the skew cubic congruence, is such that the three rays thereof, proceeding from an arbitrary point in space, are not, in general, coplanar. The properties of the latter congruence were fully developed by Kummer; whilst those of the former were only very briefly alluded to by him.

2. A year ago, in a paper communicated to the London Mathematical Society, I had occasion to study a special case of the abovementioned skew cubic congruence.† It was of the Cremonian

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* *Über diejenigen Flächen, welche mit ihren reciprok polaren Flächen von derselben Ordnung sind und die gleichen Singularitäten besitzen.*

† *On Congruences of the Third Order and Class*, "Proceedings of the London Mathematical Society," Vol. XVI., pp. 232—38, 1885.

I may here mention that, in 1882, Dr. Roccella published, at Piazza Armerina, in Sicily, an interesting thesis entitled, *Sugli enti geometrici dello spazio di retti generati dalle intersezione dè complessi correspondentí in due o più fasci proiettivi di complessi lineari*, in which, amongst other things, he speaks of a congruence of the third order and class, definable as *the locus of a right line constantly incident with three corresponding rays of three given projective pencils, arbitrarily situated in space.* This congruence, as I have recently shown, in a communication to the *Circolo Matematico di Palermo* ("Rendiconti," t. I., *seduta del* 21 *febrajo* 1886), is itself a special case of the one studied by me, and referred to in the text.

I am also informed by Prof. Sturm, of Münster, that he has been led, still more recently, and quite independently, to a somewhat similar, purely descriptive method of generating the congruence described in my paper of 1885. In place of one of the three projective pencils employed by Roccella, he simply substitutes a quadric regulus, one of whose generators coincides with its corresponding ray in one of the two remaining projective pencils.

type; that is to say, a congruence whose rays determine a Cremonian, or birational, correspondence between the points of two planes. Its investigation naturally raised the question as to the existence and generation of a Cremonian cubic congruence of Kummer's first, or non-skew type; and this enquiry, just as naturally, led to the wider one which forms the subject of the present paper.

In it I propose to consider, *first* under what conditions a congruence, contained in a linear complex, will be Cremonian (Arts. 3—7); *secondly*, how such congruences may be generated, and what varieties they present (Arts. 8—23); and, *finally*, what special properties those of the third order and class possess (Arts. 24—26).

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3. The order and class of every congruence contained in a linear complex are necessarily equal to one another; and their common value, say n (>2), also indicates, if the congruence be Cremonian, the degree of the birational correspondence whence such congruence proceeds.

The truth of the first part of this statement is sufficiently obvious; that of the second follows from the first, and from the fact that the degree of the correspondence is, by definition, the order of the curve which corresponds, in it, to a right line. This order is, in fact, clearly equal to the number of congruence-rays situated in any plane passing through that line; in other words, to the class n of the congruence (*C. C.*, Art. 3).*

4. The order of a Cremonian congruence ordinarily exceeds its class by two, and order and class only become equal to one another when, in consequence of the presence of two self-respondent points C and D, on the intersection of the generating planes α and β, two pencils of congruence-rays become detached from those which connect corresponding points. (*C. C.*, Art. 20.)

5. Only one of the n congruence-rays proceeding from an arbitrary point A of α, say, passes through the corresponding point B of β; the remaining ones connect points of β, situated on $\overline{\alpha\beta}$, with the several points in α which correspond to them. Ordinarily, the latter envelope a congruence-curve in α of the class $n-1$.

But, if the congruence under consideration form part of a linear complex, these $n-1$ rays must coincide with one another, otherwise

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* As in my last paper, I shall refer thus to my memoir *On Cremonian Congruences*, published in Vol. XIV. of the "Proceedings of the London Mathematical Society," pp. 259—301, 1883.

they could not always lie, with the n^{th} ray $\overline{AB}$, in one and the same plane. In other words, the above-mentioned congruence-curve becomes a congruence-pencil, each of whose rays is to be counted $n-1$ times; viz., once for every point of such ray, which is thereby connected with its corresponding point on $\overline{\alpha\beta}$.

Since the last-mentioned point has more than one, it must have an infinite number of corresponding points in α, all which points must be situated on a curve of the order $n-1$. It is, in fact, a principal point, B_{n-1}, of β of the same order, $n-1$, of multiplicity, and being necessarily the only point of this kind in β, it must likewise be the centre of the congruence-pencil in α above alluded to.

6. From the above analysis we readily conclude that the birational correspondence between α and β whence a Cremonian congruence contained in a linear complex proceeds must be of the isographic (*de Jonquières*) type, with its principal multiple points A_{n-1} and B_{n-1} situated on the intersection $\overline{\alpha\beta}$; to which multiple points correspond, in β and α respectively, principal curves b^{n-1} and a^{n-1} of the order $n-1$, and having B_{n-1} and A_{n-1} for $(n-2)$-ple points. Exclusive of the above principal curves, therefore, to every right line a, in α, passing through A_{n-1}, corresponds a right line b, in β, which passes through B_{n-1}. Not merely do the corresponding points of these lines a and b form two projective rows, but the lines themselves are corresponding rays of two projective pencils $A_{n-1}(\alpha)$ and $B_{n-1}(\beta)$. Of these pencils, moreover, $\overline{\alpha\beta}$ is a self-respondent ray, since it contains the two self-respondent points C and D (Art. 4).

7. Conversely, every isographic correspondence between α and β whose principal multiple points are situated on $\overline{\alpha\beta}$, and in which the latter line is self-respondent, generates a congruence which is contained in a linear complex.*

* Although not essential to the present enquiry, a brief consideration of the interesting problem, "To determine a Cremonian Congruence which shall be contained in a *given* linear complex," here merits a place.

A plane α, and in it a point A_{n-1}, being arbitrarily assumed, the centre B_{n-1} of the complex-pencil situated in α will necessarily lie in the plane β of the complex-pencil of which A_{n-1} is the centre; and every ray a of the pencil $A_{n-1}(\alpha)$ will have for "conjugate polar," relative to the given complex, a perfectly determinate ray b of the pencil $B_{n-1}(\beta)$ [see Plücker's *Neue Geometrie des Raumes*, p. 28]. Not only are a and b, however, corresponding rays of the two projective pencils $A_{n-1}(\alpha)$ and $B_{n-1}(\beta)$, having a common self-respondent complex-ray $\overline{\alpha\beta}$; they are likewise, as is well known, the directrices of a linear congruence contained in the given complex. The latter, indeed, is simply the aggregate of all such congruences.

From this it follows at once that, if an isographic correspondence of the degree n could be established between α and β, of which A_{n-1} and B_{n-1} were the multiple principal points, and every pair of rays a, b corresponding lines, the Cremonian Congruence thereby generated would satisfy the conditions of the problem.

The establishment of such an isographic correspondence presents no difficulty.

For from an arbitrary point P, in space, the corresponding rays a and b are projected by the planes of two projective pencils which have in $(P, \overline{\alpha\beta})$ a self-respondent plane. The intersections of all other pairs of corresponding planes, therefore, are coplanar; they form, in fact, a plane pencil of rays $P(\pi)$, and belong to a linear complex. Amongst them, of course, are the several rays, passing through P, of the Cremonian congruence which the assumed isographic correspondence between α and β generates.

The same result will be arrived at if the projective pencils $A_{n-1}(\alpha)$, $B_{n-1}(\beta)$ be cut by an arbitrary plane π. The corresponding rays a, b then determine on $\overline{\pi\alpha}, \overline{\pi\beta}$, two projective rows which have in $(\pi, \overline{\alpha\beta})$ a self-respondent point. The connectors of all other pairs of corresponding points, therefore, are concurrent, say in P, and in the pencil $P(\pi)$ will necessarily be found the several rays of the Cremonian congruence under consideration.

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8. If the isographic correspondence by which this congruence is generated be of the degree n, and C and D be the only two self-respondent points thereof, the congruence itself will be of the order as well as of the class n (Art. 4).

9. If, however, three, and consequently all, points of $\overline{\alpha\beta}$ be self-respondent, a special linear complex may be detached from the aggregate of rays joining corresponding points (*C. C.*, Art. 22), and there will remain a congruence $(n-1, n-1)$, which, as in the above more general case, is itself contained in a linear complex.*

10. Of Cremonian congruences contained in a linear complex we have, consequently, two distinct types. Those of the *first* type, described in Art. 8, have four singular points on $\overline{\alpha\beta}$; those of the *second*, only two, as explained in Art. 9.

11. From the mode in which the congruence (n, n) of the *first type* has been generated in Art. 6, it is obvious that it possesses two $(n-1)$-fold congruence-pencils, in the generating planes α and β, having their respective vertices B_{n-1} and A_{n-1} situated on $\overline{\alpha\beta}$. It has, moreover, $2(n-1)$ pairs of congruence-pencils whose centres are the associated principal single points of the isographic correspondence

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* This congruence may also be generated, after the manner of Roccella and Sturm (see Note to Art. 2), as the locus of a right line which is constantly incident with three corresponding generators of three projective forms; viz., two plane pencils having a self-respondent ray, and a unicursal scroll.

whence the congruence proceeds, and whose planes pass through the principal lines corresponding to those centres. An additional pair of congruence-pencils has its centres at the self-respondent points C and D, situated on $\overline{\alpha\beta}$; its planes, γ and δ, intersect each other in that same line $\overline{\alpha\beta}$. (*C. C.*, Art. 21.) The congruence itself, moreover, may be regarded as the aggregate of all the quadric reguli having, for directrices, the several pairs of corresponding rays a, b of the projective pencils $A_{n-1}(\alpha)$ and $B_{n-1}(\beta)$, and whose generators join corresponding points of these rays (Art. 6). The reguli respectively conjugate to these form, in the aggregate, another congruence (n, n) (*C. C.*, Art. 36a). This associated congruence, however, is not Cremonian.

12. The common focal surface, however, of the latter, and of the congruence (n, n) in which we are more immediately interested, is the envelope of the system of quadric surfaces upon which the several systems of conjugate reguli are situated. It is of the order as well as of the class $4(n-1)$ (*C. C.*, Arts. 10 and 20), touches each of the planes α and β along the principal curve of the order $n-1$ which that plane contains, and likewise *cuts* it along the $2(n-2)$ tangents which can be drawn to that curve from the principal multiple point to which it corresponds. The line $\overline{\alpha\beta}$, moreover, is a double one on the focal surface under consideration.

13. In support of these statements, I observe that the quadric $(a, b)^2$, and therefore the focal surface it envelopes, touches the planes α and β respectively, at the points A_0 and B_0, where its directrices a and b cut, ulteriorly, the principal curves a^{n-1} and b^{n-1}. For to these points correspond, respectively, the principal points B_{n-1} and A_{n-1}, so that $a_0 \equiv \overline{A_0 B_{n-1}}$ and $b_0 \equiv \overline{B_0 A_{n-1}}$ are generators of the quadric $(a, b)^2$. The latter, of course, likewise touches the focal surface along the quartic curve (characteristic), in which it is intersected by the next succeeding quadric of the system. Now this quartic curve clearly breaks up into a cubic and the generator a_0 or b_0 whenever the latter happens to touch the principal curve a^{n-1} or b^{n-1} at A_0 or B_0. Hence it follows that all such generators a_0 and b_0 lie wholly on the focal surface, and in the composite section of that surface, made by either of the planes α or β, they count as an element of the order $2(n-2)$; this being, in general, the class of a^{n-1} or b^{n-1}.

Each of the latter curves, moreover, being the curve of contact between its plane and the focal surface, counts as another element of the section, made with the latter by the former, of the order $2(n-1)$;

so that, the order of the total section being $4(n-1)$, the residual element thereof can only be of the order

$$4(n-1)-2(n-1)-2(n-2) = 2.$$

This proceeds from $\overline{\alpha\beta}$, which is a double line on the focal surface.

14. The self-respondent points C and D of Arts. 4 and 11 are nodes of the focal surface, at each of which the quadric cone of contact breaks up into a pair of right lines. One of these, at both points, is $\overline{\alpha\beta}$; the other, we will denote by c at C, and by d at D.

In fact, confining our attention for the present to the point C, if c' and c'' be two right lines, in α and β respectively, each of which touches, at C, the curve corresponding to the other, two of the n congruence-rays in the plane (c', c'') will coincide with the intersection of the latter and γ. (*C. C.*, Art. 19.) But c' and c'' are obviously corresponding rays of two pencils which have in $\overline{\alpha\beta}$ a self-respondent element, so that the quadric cone enveloped by the plane (c', c'') breaks up into two pencils, one of which has $\overline{\alpha\beta}$, and the other c for its axis.

15. The focal surface, like the congruence (n, n) itself, is self-reciprocal. Hence we may infer from the above that γ and δ are double planes of that surface, and that the conic of contact, in each of them, breaks up into a pair of right lines; viz., $\overline{\alpha\beta}$ and $\bar{c}$ in γ, $\overline{\alpha\beta}$ and $\bar{d}$ in δ.

16. It is worthy of note, also, that of any congruence-ray of the pencil $C(\gamma)$ or $D(\delta)$, one focus is fixed at C or D, and the other, variable with the ray, moves on the line $\bar{c}$ or $\bar{d}$; whilst one focal plane is fixed at γ or δ, and the other, variable with the ray, turns around c or d.

17. The points A_{n-1} and B_{n-1} are multiple ones on the focal surface. At each of them the cone of contact breaks up into the right line $\overline{\alpha\beta}$, axis of a pencil of planes, and a unicursal cone of the class $n-1$ which touches α [β] along the $n-2$ tangents to the principal curve a^{n-1} [b^{n-1}] at its multiple point A_{n-1} [B_{n-1}].

In fact, confining our attention for a moment to the point A_{n-1}, the cone of contact thereat is the envelope of the plane (a, b_0) (Art. 13) which touches the quadric $(a, b)^2$ at A_{n-1}. Now, to each ray a, in α, corresponds one, and only one, ray b_0, in β; whilst to each ray b_0 —since it cuts b^{n-1} in $n-1$ points B_0, to each of which proceeds a ray b —correspond $n-1$ rays a. Of this $(1, n-1)$ correspondence between the rays a and b_0, however, $\overline{\alpha\beta}$ is a self-respondent element, so that,

by a well-known theorem, the plane (a, b_0) of two corresponding rays envelopes a cone, of the class n, which breaks up into the pencil of planes whose axis is $\overline{\alpha\beta}$, and a cone of the class $n-1$ having α for a $(n-2)$-ple tangent plane. The generators of contact with the latter plane are the rays a which touch a^{n-1} at its $(n-2)$-ple point A_{n-1} (Art. 12); since these correspond, as may be easily verified, to the $(n-2)$ rays b touching b^{n-1} at its multiple point B_{n-1}, with which latter $n-2$ of the points B_0 coincide when b_0 falls on $\overline{\alpha\beta}$.

18. From Plücker's formulæ we conclude, further, that the cones of contact at A_{n-1} and B_{n-1}, which, as we have just seen, are of the class $n-1$, and of the order $2(n-2)$, have each $3(n-3)$ cuspidal edges, and $2(n-3)(n-4)$ double ones.

These cuspidal edges, it may be observed, are tangents at A_{n-1} and B_{n-1} to a curve of regression,* and in like manner we may infer that the double edges, above alluded to, give the directions, at the last named points, of a double curve on the focal surface.

19. It is scarcely necessary to add that the $2(n-1)$ pairs of associated principal single points A_1, B_1 of the isographic correspondence between α and β are also nodes of the focal surface; at these points the quadric cones of contact are both touched by the planes $(A_1B_1B_{n-1})$ and $(B_1A_1A_{n-1})$; which planes, moreover, are singular tangent planes of the focal surface; that is to say, each touches the latter along a conic which passes through A_1 as well as B_1.†

20. Proceeding, now, to the Cremonian congruences of the *second type* (Art. 10), which are contained in a linear complex, and whose order and class are $n-1$, when the generating isographic correspondence is of the degree n (Art. 9), I observe that α and β contain congruence-pencils whose centres are B_{n-1} and A_{n-1} respectively, and with every ray of which $n-2$, in place of $n-1$, congruence-rays coincide. This difference, it need scarcely be said, arises from the fact that the principal curves a^{n-1} and b^{n-1} now pass, respectively, through B_{n-1} and A_{n-1}, so that, exclusive of the latter, they are only cut in $n-2$ other points by every ray of the pencils $B_{n-1}(\alpha)$ and $A_{n-1}(\beta)$.

In addition to these $(n-2)$-fold pencils, the congruence $(n-1, n-1)$ now under consideration possesses $2(n-1)$ pairs of ordinary ones. The centres of each pair are associated principal single points, and their planes pass respectively through the principal lines corresponding to those centres.

* See Arts. 12 and 13 of my paper, *On Congruences of the Third Order and Class*, "Proceedings of the London Mathematical Society," Vol. XVI., p. 235, 1885.

† *Ibid.*, Art. 10.

The congruence itself is again the aggregate of all quadric reguli whose directrices are corresponding rays a and b of the pencils $A_{n-1}(\alpha)$ and $B_{n-1}(\beta)$. When these directrices are coincident in $\overline{\alpha\beta}$, however, their corresponding points likewise coincide, so that the regulus no longer degenerates, as in Art. 11, to a pair of pencils $C(\gamma)$, $D(\delta)$. It is easy to see, for instance, that the generators a_0 and b_0 of this regulus, which lie in the planes α and β respectively, are the tangents at B_{n-1} and A_{n-1} of the principal curves a^{n-1} and b^{n-1}.

21. The focal surface of our congruence $(n-1, n-1)$ is again the envelope of the several quadrics $(a, b)^2$ on which the above reguli are situated. Its order and class, however, are now $4(n-2)$.* It touches α and β, as before, along the principal curves a^{n-1} and b^{n-1}, which latter, it must be remembered, have not only $(n-2)$-ple points at A_{n-1} and B_{n-1}, but, as stated in Art. 20, also pass, respectively, through B_{n-1} and A_{n-1}. This focal surface likewise *cuts* the planes α and β along the $2(n-3)$ tangents which can be drawn from B_{n-1} and A_{n-1} respectively, to touch *elsewhere* the principal curves a^{n-1} and b^{n-1}. The section of the focal surface with each of the planes α and β is thus seen to be of the already-stated order, viz.:

$$2(n-1)+2(n-3) = 4(n-2).$$

22. The singularities of the points A_{n-1} and B_{n-1} on the focal surface are again precisely correlative to those of the planes α and β. At each of these multiple points the cone of contact is of the class $n-1$. That with vertex at $A_{n-1}[B_{n-1}]$ touches the plane $\alpha[\beta]$ $n-2$ times, and $\beta[\alpha]$ once; both, in fact, along the tangents to the principal curves which pass through $A_{n-1}[B_{n-1}]$. This cone, moreover, besides *touching* the plane $\beta[\alpha]$ once, as above stated, *cuts* it along the $2(n-3)$ tangents that can be drawn from $A_{n-1}[B_{n-1}]$ to touch the principal curve $b^{n-1}[a^{n-1}]$ elsewhere. In fact,

$$2+2(n-3) = 2(n-2)$$

is, in general, the order of the cone in question.

The singularities described in Art. 18 reappear, without change, in the cones now under consideration.

23. The singularities presented by the focal surface at the $2(n-1)$ pairs of associated principal single points A_1 and B_1 are precisely the same as those described in Art. 19. They may, however, be elucidated somewhat differently, thus:—

The system of quadrics $(a, b)^2$ includes $2(n-1)$ point-and-plane

* *C. C.*, Art. 23, in which m is to be replaced by unity.

pairs. The points of each, A_1, B_1, are associated principal single points; the planes of each $(A_1B_1B_{n-1})$, $(B_1A_1A_{n-1})$ connect these points with the principal lines which respectively correspond to them. These planes are cut by the next succeeding quadric $(a, b)^2$ of the system in a pair of conics which intersect in A_1 and B_1; they are, in fact, singular tangent planes of the focal surface, and these are their conics of contact therewith. Correlatively A_1 and B_1 are nodes of the focal surface, the quadric cones of contact at which touch both the singular planes just referred to.

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24. In accordance with the scheme proposed in Art. 2, I now pass to a brief consideration of the special properties of the two different Cremonian congruences of the third order and class which are contained in a linear complex. The first is obtained by putting $n=3$ in Arts. 11—19; the second, by making $n=4$ in Arts. 20—23. For both these congruences, it will be observed (Arts. 12 and 21), the focal surface is of the eighth order and class; in other respects, however, the congruences in question differ from each other materially.

25. That of the *first type* has doubled congruence-pencils in the planes α and β, with centres at B_2 and A_2, respectively, on $\overline{\alpha\beta}$. It has also four pairs of congruence-pencils whose centres A_1 and B_1 are associated principal single points of the cubic correspondence between α and β, and whose planes $(A_1B_1B_2)$ and $(B_1A_1A_2)$ pass through the principal lines corresponding to those centres. It has, moreover, a fifth pair of congruence-pencils whose centres are the self-respondent points C and D, and whose respective planes γ and δ pass through the intersection $\overline{\alpha\beta}$.

The focal surface has $\overline{\alpha\beta}$ for a double line, and the five pairs of singular points and planes above enumerated have, for it, precisely the properties described in Arts. 14, 15, 16, and 19.

This focal surface touches α and β along the principal *conics* passing respectively through A_2 and B_2, and it cuts these planes, moreover, along the tangents to these conics which proceed from B_2 and A_2 respectively.

Correlatively, the points A_2 and B_2 are nodes on the focal surface; at each of which the cone of contact breaks up into $\overline{\alpha\beta}$, regarded as the axis of a pencil of planes, and a cone of the second class. The cone whose vertex is at A_2 [B_2], for instance, not only touches α [β] along the tangent at A_2 [B_2] to the principal conic, but it likewise cuts β [α] along the two tangents from A_2 [B_2] to the principal conic in the latter plane. These tangents, as we have just seen, lie wholly on the focal surface.

26. The congruence (3, 3) of the *second type* has, like that of the first type, doubled congruence-pencils in the planes α and β, the centres of which are at B_3 and A_3 respectively. But, instead of having five, it has six pairs of congruence-pencils, the vertices of which are all associated principal single points of the quartic, isographic correspondence whence the congruence proceeds, and the planes of which pass, as usual, through the principal lines corresponding to those points.

On the focal surface, these points and planes are singular ones of the kind already described in Art. 23 and elsewhere, and the surface in question touches the planes α and β along the principal cubics which these planes contain. Besides touching α [β], however, along this cubic, which has a double point at A_3 [B_3] and passes through B_3 [A_3], it cuts it along the two tangents to this cubic which can be drawn from B_3 [A_3] to touch the curve elsewhere.

Correlatively, A_3 and B_3 are nodes on the focal surface, at which the cones of contact are of the third class and fourth order. Of each cone, one of the planes α and β is a double, and the other an ordinary tangent plane. The cone whose vertex is A_3, for instance, touches α along the tangents at the double point A_3 of the principal cubic in α, and it also touches β along the tangent at A_3 to the principal cubic in β. At the same time it cuts the latter plane along the remaining two tangents that can be drawn from A_3 to the cubic just referred to. The last-mentioned generators of the cone of contact at its singular point A_3 lie, indeed, wholly on the focal surface.

On the Airy-Maxwell Solution of the Equations of Equilibrium of an Isotropic Elastic Solid, under Conservative Forces. By W. J. Ibbetson, M.A., F.R.A.S.

[*Read May 13th*, 1886.]

Sir G. B. Airy was the first to propose* a very elegant method of solving the equations of stress in two dimensions, the very obvious extension of which to three dimensions is due to Clerk Maxwell.†

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* *British Association Report*, Cambridge, 1862, p. 82; and *Phil. Trans.* for 1863, p. 49.
† *Edinburgh Trans.*, Vol. xxvi., p. 31.

Airy himself seems to have regarded the function upon which his solution depends as entirely arbitrary in form; and Clerk Maxwell, after pointing out the inconsistency of Airy's results with the general conditions of strain, passes very lightly over the limitations to which the method is subject, and himself gives a solution which equally fails to satisfy those conditions.

I propose, in the present paper, to start with the general equations of equilibrium in three dimensions; to deduce Maxwell's formulæ for the component stresses; and, by applying the conditions of integrability, to obtain the corresponding expressions for the components of displacement. A similar mode of treatment applied to the equations of plane stress will lead us without difficulty to the most general form of solution, applicable to Airy's case of rectangular beams under gravity. We shall see that the method is available only for a limited class of cases, and it is possible that some of those discussed by Airy are altogether beyond its scope.

If ρ denote the natural density of the body, Ψ the potential of the applied forces per unit mass, and P, Q, R, S, T, U the normal and tangential components of stress, the general equations of equilibrium may be written

$$\left.\begin{aligned} \frac{d(P+\rho\Psi)}{dx}+\frac{dU}{dy}+\frac{dT}{dz}&=0 \\ \frac{dU}{dx}+\frac{d(Q+\rho\Psi)}{dy}+\frac{dS}{dz}&=0 \\ \frac{dT}{dx}+\frac{dS}{dy}+\frac{d(R+\rho\Psi)}{dz}&=0 \end{aligned}\right\} \quad \text{.....................(1)},$$

and it is at once evident, on substitution, that these are satisfied by the assumptions

$$\left.\begin{aligned} P&=\frac{d^2\chi_3}{dy^2}+\frac{d^2\chi_2}{dz^2}-\rho\Psi \\ Q&=\frac{d^2\chi_1}{dz^2}+\frac{d^2\chi_3}{dx^2}-\rho\Psi \\ R&=\frac{d^2\chi_2}{dx^2}+\frac{d^2\chi_1}{dy^2}-\rho\Psi \\ S&=-\frac{d^2\chi_1}{dy\,dz} \\ T&=-\frac{d^2\chi_2}{dz\,dx} \\ U&=-\frac{d^2\chi_3}{dx\,dy} \end{aligned}\right\} \quad \text{.......................(2)},$$

where χ_1, χ_2, χ_3 may *so far* be any continuous single-valued functions of position. It is, however, obvious that the *six* stress components, being linear functions of the first derivatives of the *three* independent components of displacement, must satisfy *six* independent differential equations of the second order, in order to insure the possibility of re-integration.

These differential equations are most easily obtained as follows.

Let e, f, g; a, b, c; $\theta_1, \theta_2, \theta_3$ be the components of dilatation, shear, and rotation, respectively; then we may show without difficulty that

$$\left.\begin{aligned} &2\frac{d\theta_1}{dx} = \frac{db}{dy} - \frac{dc}{dz}, \quad 2\frac{d\theta_1}{dy} = \frac{da}{dy} - 2\frac{df}{dz}, \quad 2\frac{d\theta_1}{dz} = 2\frac{dg}{dy} - \frac{da}{dz} \\ &2\frac{d\theta_2}{dx} = 2\frac{de}{dz} - \frac{db}{dx}, \quad 2\frac{d\theta_2}{dy} = \frac{dc}{dz} - \frac{da}{dx}, \quad 2\frac{d\theta_2}{dz} = \frac{db}{d6} - 2\frac{dg}{dx} \\ &2\frac{d\theta_3}{dx} = \frac{dc}{dx} - 2\frac{de}{dy}, \quad 2\frac{d\theta_3}{dy} = 2\frac{df}{dx} - \frac{dc}{dy}, \quad 2\frac{d\theta_3}{dz} = \frac{da}{dx} - \frac{db}{dy} \end{aligned}\right\} \ldots(3).$$

On eliminating $\theta_1, \theta_2, \theta_3$ from equations (3) by cross differentiation in all possible ways, we obtain

$$\left.\begin{aligned} \frac{d^2g}{dy^2} + \frac{d^2f}{dz^2} &= \frac{d^2a}{dy\,dz} \\ \frac{d^2e}{dz^2} + \frac{d^2g}{dx^2} &= \frac{d^2b}{dz\,dx} \\ \frac{d^2f}{dx^2} + \frac{d^2e}{dy^2} &= \frac{d^2c}{dx\,dy} \\ 2\left(\frac{d^2e}{dy\,dz} + \frac{d^2a}{dx^2}\right) &= \frac{d}{dx}\left(\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz}\right) \\ 2\left(\frac{d^2f}{dz\,dx} + \frac{d^2b}{dy^2}\right) &= \frac{d}{dy}\left(\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz}\right) \\ 2\left(\frac{d^2g}{dx\,dy} + \frac{d^2c}{dz^2}\right) &= \frac{d}{dz}\left(\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots(4).$$

These are the six equations required, and they must be satisfied *identically* by every system of values that can be legitimately assumed for the component strains.

If q denotes Young's modulus, and σ the ratio of lateral contraction

to longitudinal elongation under simple tension,

$$\left.\begin{array}{l} e = P/q - \sigma(Q+R)/q \\ a = 2(1+\sigma)\,S/q \\ \quad \&c. \qquad \&c. \end{array}\right\}.$$

Thus, on substitution from (2), the values of the strain components, expressed in terms of the χ-functions, are given by

$$\left.\begin{array}{l}
qe = \Phi - (1+\sigma)\left[\nabla^2\chi_1 + \dfrac{d^2}{dx^2}(\chi_2+\chi_3-\chi_1)\right] \\
qf = \Phi - (1+\sigma)\left[\nabla^2\chi_2 + \dfrac{d^2}{dy^2}(\chi_3+\chi_1-\chi_2)\right] \\
qg = \Phi - (1+\sigma)\left[\nabla^2\chi_3 + \dfrac{d^2}{dz^2}(\chi_1+\chi_2-\chi_3)\right] \\
qa = -2(1+\sigma)\dfrac{d^2\chi_1}{dy\,dz} \\
qb = -2(1+\sigma)\dfrac{d^2\chi_2}{dz\,dx} \\
qc = -2(1+\sigma)\dfrac{d^2\chi_3}{dx\,dy}
\end{array}\right\} \quad \ldots\ldots\ldots\ldots(5),$$

where $\Phi = \nabla^2(\chi_1+\chi_2+\chi_3) - \left(\dfrac{d^2\chi_1}{dx^2} + \dfrac{d^2\chi_2}{dy^2} + \dfrac{d^2\chi_3}{dz^2}\right) - (1-2\sigma)\,\rho\Psi \quad \ldots(6).$

Inserting these values in equations (4), they reduce to

$$\left.\begin{array}{l}
\dfrac{d^2}{dy^2}\left[\Phi - (1+\sigma)\nabla^2\chi_3\right] + \dfrac{d^2}{dz^2}\left[\Phi - (1+\sigma)\nabla^2\chi_2\right] = 0 \\
\dfrac{d^2}{dz^2}\left[\Phi - (1+\sigma)\nabla^2\chi_1\right] + \dfrac{d^2}{dx^2}\left[\Phi - (1+\sigma)\nabla^2\chi_3\right] = 0 \\
\dfrac{d^2}{dx^2}\left[\Phi - (1+\sigma)\nabla^2\chi_2\right] + \dfrac{d^2}{dy^2}\left[\Phi - (1+\sigma)\nabla^2\chi_1\right] = 0 \\
\dfrac{d^2}{dy\,dz}\left[\Phi - (1+\sigma)\nabla^2\chi_1\right] = 0 \\
\dfrac{d^2}{dz\,dx}\left[\Phi - (1+\sigma)\nabla^2\chi_2\right] = 0 \\
\dfrac{d^2}{dx\,dy}\left[\Phi - (1+\sigma)\nabla^2\chi_3\right] = 0
\end{array}\right\} \quad \ldots\ldots(7),$$

thus giving six equations of the fourth order to be satisfied *identically* by χ_1, χ_2, χ_3. If each of the first three of these be integrated twice,

we obtain the more useful forms

$$\left.\begin{aligned}
&\frac{d}{dy}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_3\right]dz+\phi_1(y)\right\} \\
&\qquad+\frac{d}{dz}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_2\right]dy+\psi_1(z)\right\}=0 \\
&\frac{d}{dz}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_1\right]dx+\phi_2(z)\right\} \\
&\qquad+\frac{d}{dx}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_3\right]dz+\psi_2(x)\right\}=0 \\
&\frac{d}{dx}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_2\right]dy+\phi_3(x)\right\} \\
&\qquad+\frac{d}{dy}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_1\right]dx+\psi_3(y)\right\}=0
\end{aligned}\right\}\quad\ldots\ldots(8),$$

where the complementary functions ϕ, ψ are wholly arbitrary.

We can now integrate the component rotations; for, on substituting from (5) in (3), we have

$$\left.\begin{aligned}
&\frac{d}{dx}\left[q\theta_1+(1+\sigma)\frac{d^2(\chi_2-\chi_3)}{dy\,dz}\right]=0 \\
&\frac{d}{dy}\left[q\theta_1+(1+\sigma)\frac{d^2(\chi_2-\chi_3)}{dy\,dz}\right]=-\frac{d}{dz}\left[\Phi-(1+\sigma)\,\nabla^2\chi_2\right] \\
&\frac{d}{dz}\left[q\theta_1+(1+\sigma)\frac{d^2(\chi_2-\chi_3)}{dy\,dz}\right]=+\frac{d}{dy}\left[\Phi-(1+\sigma)\,\nabla^2\chi_3\right]
\end{aligned}\right\}$$

whence, by the help of (8),

$$\left.\begin{aligned}
q\theta_1+(1+\sigma)\frac{d^2(\chi_2-\chi_3)}{dy\,dz}&=\frac{d}{dy}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_3\right]dz+\phi_1(y)\right\} \\
&=-\frac{d}{dz}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_2\right]dy+\psi_1(z)\right\} \\
q\theta_2+(1+\sigma)\frac{d^2(\chi_3-\chi_1)}{dz\,dx}&=\frac{d}{dz}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_1\right]dx+\phi_2(z)\right\} \\
&=-\frac{d}{dx}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_3\right]dz+\psi_2(x)\right\} \\
q\theta_3+(1+\sigma)\frac{d^2(\chi_1-\chi_2)}{dx\,dy}&=\frac{d}{dx}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_2\right]dy+\phi_3(x)\right\} \\
&=-\frac{d}{dy}\left\{\int\left[\Phi-(1+\sigma)\,\nabla^2\chi_1\right]dx+\psi_3(y)\right\}
\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots(9).$$

Now, if u, v, w be the component displacements, it is easy to show that
$$\frac{du}{dx} = e, \quad \frac{du}{dy} = \tfrac{1}{2}c - \theta_3, \quad \frac{du}{dz} = \tfrac{1}{2}b + \theta_2 \quad \ldots\ldots\ldots\ldots\ldots(10);$$

and therefore, by (9),

$$\left.\begin{aligned}
\frac{d}{dx}\left\{qu - \int\left[\Phi-(1+\sigma)\,\nabla^2\chi_1\right]dx + (1+\sigma)\frac{d}{dx}(\chi_2+\chi_3-\chi_1)\right\} &= 0\\
\frac{d}{dy}\left\{qu - \int\left[\Phi-(1+\sigma)\,\nabla^2\chi_1\right]dx + (1+\sigma)\frac{d}{dx}(\chi_2+\chi_3-\chi_1)\right\} &= \psi_3'(y)\\
\frac{d}{dz}\left\{qu - \int\left[\Phi-(1+\sigma)\,\nabla^2\chi_1\right]dx + (1+\sigma)\frac{d}{dx}(\chi_2+\chi_3-\chi_1)\right\} &= \phi_2'(z)
\end{aligned}\right\},$$

and, finally,

$$\left.\begin{aligned}
qu &= \int\left[\Phi-(1+\sigma)\,\nabla^2\chi_1\right]dx - (1+\sigma)\frac{d}{dx}(\chi_2+\chi_3-\chi_1) + \psi_3(y) + \phi_2(z)\\
qv &= \int\left[\Phi-(1+\sigma)\,\nabla^2\chi_2\right]dy - (1+\sigma)\frac{d}{dy}(\chi_3+\chi_1-\chi_2) + \psi_1(z) + \phi_3(x)\\
qw &= \int\left[\Phi-(1+\sigma)\,\nabla^2\chi_3\right]dz - (1+\sigma)\frac{d}{dz}(\chi_1+\chi_2-\chi_3) + \psi_2(x) + \phi_1(y)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(11).$$

In the case of *Plane Stress* (*e.g.*, in planes perpendicular to Oz), we have $R = S = T = 0$, while P, Q, U, and Ψ are independent of z. On substitution of these quantities in the identical equations (4), the fourth and fifth are satisfied identically, while the others reduce to

$$\left.\begin{aligned}
\frac{d^2}{dx^2}(P+Q) = \frac{d^2}{dy^2}(P+Q) = \frac{d^2}{dx\,dy}(P+Q) &= 0\\
\frac{d^2P}{dx^2} + \frac{d^2Q}{dy^2} + 2\,\frac{d^2U}{dx\,dy} = \frac{1}{1+\sigma}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)(P+Q) &= 0
\end{aligned}\right\} \ldots (12).$$

But, by differentiating the first of equations (1) as to x, and the second as to y, and adding the results, we obtain

$$\frac{d^2P}{dx^2} + \frac{d^2Q}{dy^2} + 2\,\frac{d^2U}{dx\,dy} = -\rho\left(\frac{d^2\Psi}{dx^2}+\frac{d^2\Psi}{dy^2}\right);$$

and, on substitution in the last of equations (12), we see that

$$\frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dy^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

It appears, then, that in all cases of Plane Stress the sum of the principal stresses must be a linear function of the coordinates, and that equilibrium is impossible unless the force potential satisfies (13).

We can now determine the form of the stress functions without difficulty. Let ∇^2 represent the operator $d^2/dx^2+d^2/dy^2$, and let ψ be the particular integral of the equation

$$\nabla^2\psi = \Psi \qquad (14),$$

then (12) and (1) give us

$$\left.\begin{aligned}\nabla^2\left(P-2\rho\,\frac{d^2\psi}{dy^2}\right) &= 0\\ \nabla^2\left(Q+2\rho\,\frac{d^2\psi}{dy^2}\right) &= 0\\ \nabla^2\left(U+2\rho\,\frac{d^2\psi}{dx\,dy}\right) &= 0\end{aligned}\right\}.$$

The appropriate solution of these equations is obviously

$$\left.\begin{aligned}P &= ax+\beta y+\gamma+2\rho\,\frac{d^2\psi}{dy^2}+\eta\\ Q &= ax+\beta y+\gamma-2\rho\,\frac{d^2\psi}{dy^2}-\eta\\ U &= -2\rho\,\frac{d^2\psi}{dx\,dy}+\xi\end{aligned}\right\},$$

where a, β, γ are arbitrary constants, and ξ, η are any solutions of (13). Substituting these values in (1), we have

$$\left.\begin{aligned}\frac{d}{dx}(\eta+\rho\Psi+ax-\beta y)+\frac{d\xi}{dy} &= 0\\ \frac{d\xi}{dx}-\frac{d}{dy}(\eta+\rho\Psi+ax-\beta y) &= 0\end{aligned}\right\},$$

so that ξ and $\eta+\rho\Psi+ax-\beta y$ are conjugate functions of x and y. We may therefore write

$$\left.\begin{aligned}\xi &= -\beta x-ay-\frac{d^2\zeta}{dx\,dy}\\ \eta &= -\rho\Psi-\frac{d^2\zeta}{dx^2}\end{aligned}\right\},$$

where ζ is any solution whatever of (13), and finally,

$$\left.\begin{aligned} P &= \alpha x+\beta y+\gamma-\rho\Psi+2\rho\frac{d^2\psi}{dy^2}+\frac{d^2\zeta}{dy^2} \\ Q &= \alpha x+\beta y+\gamma-\rho\Psi+2\rho\frac{d^2\psi}{dx^2}+\frac{d^2\zeta}{dx^2} \\ U &= -\beta x-\alpha y-2\rho\frac{d^2\psi}{dx\,dy}-\frac{d^2\zeta}{dx\,dy} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(15).$$

Comparing our result with (2), we see that it is a particular case of the former solution, in which $\chi_1 = \chi_2 = 0$, and

$$\chi_3 = \tfrac{1}{6}\left[\alpha\,(x^3+3xy^2)+\beta\,(3x^2y+y^3)+3\gamma\,(x^2+y^2)\right]+2\rho\psi+\zeta.$$

The expressions for the displacements are most easily found by direct integration of (15), which may be written in the form

$$\left.\begin{aligned} &\frac{d}{dx}\left[qu+(1+\sigma)\frac{d}{dx}\,(2\rho\psi+\zeta)+(1+\sigma)\,\alpha y^2\right] \\ &\qquad=\frac{d}{dy}\left[qv+(1+\sigma)\frac{d}{dy}\,(2\rho\psi+\zeta)+(1+\sigma)\,\beta x^2\right] \\ &\qquad=(1-\sigma)(\alpha x+\beta y+\gamma)+(1+\sigma)\,\rho\Psi \\ &\frac{d}{dy}\left[qu+(1+\sigma)\frac{d}{dx}\,(2\rho\psi+\zeta)+(1+\sigma)\,\alpha y^2\right] \\ &\qquad+\frac{d}{dx}\left[qv+(1+\sigma)\frac{d}{dy}\,(2\rho\psi+\zeta)+(1+\sigma)\,\beta x^2\right]=0 \\ &\frac{dw}{dz}=-2\sigma\,(\alpha x+\beta y+\gamma) \end{aligned}\right\},$$

showing that the expressions in square brackets are conjugate functions of x and y.

Denoting them by ξ', η', we have

$$\frac{d\xi'}{dx}=\frac{d\eta'}{dy}=(1-\sigma)(\alpha x+\beta y+\gamma)+(1+\sigma)\,\rho\Psi\,;$$

whence, if
$$\frac{d}{dx}\int\Psi\,dy+\frac{d}{dy}\int\Psi\,dx = X+Y \quad\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

X being a function of x only, and Y of y only, by reason of (13),

$$\left.\begin{aligned} qu &= (1-\sigma)\left[\tfrac{1}{2}\alpha\,(x^2-y^2)+\beta xy+\gamma x\right]+\sigma\alpha z^2 \\ &\quad-(1+\sigma)\left[\alpha y^2+\rho\left(2\frac{d\psi}{dx}+\int Y\,dy-\int\Psi\,dx\right)+\frac{d\zeta}{dx}\right] \\ qv &= (1-\sigma)\left[\alpha xy+\tfrac{1}{2}\beta\,(y^2-x^2)+\gamma y\right]+\sigma\beta z^2 \\ &\quad-(1+\sigma)\left[\beta x^2+\rho\left(2\frac{d\psi}{dy}+\int X\,dx-\int\Psi\,dy\right)+\frac{d\zeta}{dy}\right] \\ qw &= -2\sigma z\left[\alpha x+\beta y+\gamma\right] \end{aligned}\right\}\quad\ldots(17).$$

Equations (17), (16), and (15) present the complete Airian solution of the problem of Plane Stress in its most general form.

In the case of rectangular beams under gravity, if Oy be directed vertically downwards, $\Psi = \mathrm{g}\, y$, and consequently $\psi = \frac{1}{12}\mathrm{g}\,(3x^2y+y^3)$, $X = \mathrm{g}\,\rho x$, $Y=0$. Thus,

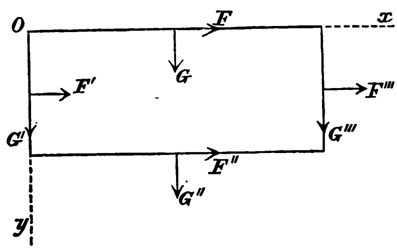
$$\left.\begin{aligned} P &= ax+\beta y+\gamma+\frac{d^2\zeta}{dy^2} \\ Q &= ax+\beta y+\gamma+\frac{d^2\zeta}{dx^2} \\ U &= -(\beta+\mathrm{g}\rho)\,x-ay-\frac{d^2\zeta}{dx\,dy} \end{aligned}\right\} \quad \text{..................(18)},$$

and

$$\left.\begin{aligned} qu &= (1-\sigma)\left[\tfrac{1}{2}a\,(x^2-y^2)+\beta xy+\gamma x\right] \\ &\qquad -(1+\sigma)\left[ay^2+\frac{d\zeta}{dx}\right]+\sigma a z^2 \\ qv &= (1-\sigma)\left[axy+\tfrac{1}{2}\beta\,(y^2-x^2)+\gamma y\right] \\ &\qquad -(1+\sigma)\left[(\beta+\mathrm{g}\rho)\,x^2+\frac{d\zeta}{dy}\right]+\sigma\beta z^2 \\ qw &= -2\sigma z\left[ax+\beta y+\gamma\right] \end{aligned}\right\} \quad \text{............(19)}.$$

It will be observed that neither Airy's solutions, nor that given by Maxwell at the end of the paper referred to, are of the required form.

The complete solution, which requires the determination of ζ and the arbitrary constants a, β, γ, is easily obtained when the distribution of the surface stress or surface displacement is completely known. We proceed to solve the former problem in general terms.



Let the upper edge of one end of the beam be taken as axis of z, the axis of y being directed vertically downwards, and the axis of x horizontally in the direction of the length of the beam. Let L be the length, and D the vertical depth. Then we are to suppose that, when $y=0$ and when $y=D$, Q and U are known for all values of x between and including 0 and L; and similarly, when $x=0$, or L, P, and U are known, from $y=0$ to $y=D$ inclusive. Let these surface stresses be denoted by the letters F and G, as in

Thomson and Tait's notation, and let accents distinguish the different faces of the beam. Then (see figure),

when $y=0$, $U=-F$, $Q=-G$, $P=2(ax+\gamma)+G$,

" $y=D$, $U=F''$, $Q=G''$, $P=2(ax+\beta D+\gamma)-G''$,

" $x=0$, $U=-G'$, $P=-F'$, $Q=2(\beta y+\gamma)+F'$,

" $x=L$, $U=G'''$, $P=F'''$, $Q=2(aL+\beta y+\gamma)-F'''$.

Now, it appears from (18) that

$$U+(\beta+\mathrm{g}\rho)x+ay \quad \text{and} \quad P-ax-\beta y-\gamma$$

are conjugate functions of x and y; so that, if U and V be conjugates,

$$P=V+(2\beta+\mathrm{g}\rho)\,y+\gamma.$$

Thus we have

when $y=0$, $U=-F$, $V=2ax+\gamma+G$,

" $y=D$, $U=F''$, $V=2ax+\gamma-\mathrm{g}\rho D-G''$,

" $x=0$, $U=-G'$, $V=-(2\beta+\mathrm{g}\rho)\,y-\gamma-F'$,

" $x=L$, $U=G'''$, $V=-(2\beta+\mathrm{g}\rho)\,y-\gamma+F'''$.

It is known that any function of x is completely represented, for all values of x between *and including* the limits 0 and L, by *either* of the series

$$\phi(x)=\phi(0)+\frac{x}{L}\left[\phi(L)-\phi(0)\right]$$

$$+\frac{2}{L}\sum_{i=1}^{i=\infty}\sin\frac{i\pi x}{L}\int_0^L\left\{\phi(p)-\phi(0)-\frac{p}{L}\left[\phi(L)-\phi(0)\right]\right\}\sin\frac{i\pi p}{L}\,dp \qquad \text{............(20)},$$

$$=\frac{1}{L}\int_0^L\phi(p)\,dp+\frac{2}{L}\sum_{i=1}^{i=\infty}\cos\frac{i\pi x}{L}\int_0^L\phi(p)\cos\frac{i\pi p}{L}\,dp \qquad \text{.........(21)},$$

both of which are perfectly determinate when the form of the function ϕ is given. We shall suppose the tangential components of the surface stresses to be expanded in the form (20), and the normal

components in the form (21), giving

$$\left.\begin{aligned}
F &= -U(0,0)+\frac{x}{L}\big[U(0,0)-U(L,0)\big]+\Sigma\mathfrak{F}_i\sin\frac{i\pi x}{L}\\
G &= \mathfrak{g}_0+\Sigma\mathfrak{g}_i\cos\frac{i\pi x}{L}\\
F' &= \mathfrak{f}'_0+\Sigma\mathfrak{f}'_i\cos\frac{i\pi y}{D}\\
G' &= -U(0,0)+\frac{y}{D}\big[U(0,0)-U(0,D)\big]+\Sigma\mathfrak{G}'_i\sin\frac{i\pi y}{D}\\
F'' &= U(0,D)+\frac{x}{L}\big[U(L,D)-U(0,D)\big]+\Sigma\mathfrak{F}''_i\sin\frac{i\pi x}{L}\\
G'' &= \mathfrak{g}''_0+\Sigma\mathfrak{g}''_i\cos\frac{i\pi x}{L}\\
F''' &= \mathfrak{f}'''_0+\Sigma\mathfrak{f}'''_i\cos\frac{i\pi y}{D}\\
G''' &= U(L,0)+\frac{y}{D}\big[U(L,D)-U(L,0)\big]+\Sigma\mathfrak{G}'''_i\sin\frac{i\pi y}{D}
\end{aligned}\right\}\ldots(22),$$

where the Old English letters denote absolute and determinate constants.

The value of the shearing stress U throughout the beam is then completely represented by the expression

$$\begin{aligned}
U = U(0,0) &+ \frac{x}{L}\big[U(L,0)-U(0,0)\big]+\frac{y}{D}\big[U(0,D)-U(0,0)\big]\\
&+\frac{xy}{LD}\big[U(L,D)+U(0,0)-U(0,D)-U(L,0)\big]\\
&+\Sigma\left[\mathfrak{F}''_i\sinh\frac{i\pi y}{L}-\mathfrak{F}_i\sinh\frac{i\pi(D-y)}{L}\right]\sin\frac{i\pi x}{L}\Big/\sinh\frac{i\pi D}{L}\\
&+\Sigma\left[\mathfrak{G}'''_i\sinh\frac{i\pi x}{D}-\mathfrak{G}'_i\sinh\frac{i\pi(L-x)}{D}\right]\sin\frac{i\pi y}{D}\Big/\sinh\frac{i\pi L}{D}
\end{aligned}\qquad\ldots\ldots\ldots\ldots\ldots(23),$$

for this satisfies $d^2U/dx^2+d^2U/dy^2=0$, and has the required values over all the boundaries. From (23) we easily deduce

$$\begin{aligned}
V = C &+ \frac{y}{L}\big[U(L,0)-U(0,0)\big]-\frac{x}{D}\big[U(0,D)-U(0,0)\big]\\
&+\frac{y^2-x^2}{2LD}\big[U(L,D)+U(0,0)-U(0,D)-U(L,0)\big]
\end{aligned}$$

$$+\Sigma\left[\mathfrak{F}_i''\cosh\frac{i\pi y}{L}+\mathfrak{F}_i\cosh\frac{i\pi(D-y)}{L}\right]\cos\frac{i\pi x}{L}\Big/\sinh\frac{i\pi D}{L}$$

$$-\Sigma\left[\mathfrak{G}_i'''\cosh\frac{i\pi x}{D}+\mathfrak{G}_i'\cosh\frac{i\pi(L-x)}{D}\right]\cos\frac{i\pi y}{D}\Big/\sinh\frac{i\pi L}{D}$$

$$\dots\dots\dots\dots(24),$$

where C is an arbitrary constant. We will postpone for the present the investigation of the relations that must exist between the components of the surface stresses, in order that this form of V may assume the required values at the boundaries; and proceed at once to determine ζ from the equations

$$\left.\begin{aligned}\frac{d^2\zeta}{dx\,dy}&=-U-(\beta+g\rho)\,x-ay\\ \frac{d^2\zeta}{dx^2}=-\frac{d^2\zeta}{dy^2}&=-V-(\beta+g\rho)\,y+ax\end{aligned}\right\}.$$

Substituting the values of U and V, and integrating,

$$\frac{d\zeta}{dx}=A-Cx-yU(0,0)+\frac{x^2-y^2}{2D}\left[aD+U(0,D)-U(0,0)\right]$$

$$-\frac{xy}{L}\left[(\beta+g\rho)\,L+U(L,0)-U(0,0)\right]$$

$$+\frac{x^3-3xy^2}{6LD}\left[U(L,D)+U(0,0)-U(0,D)-U(L,0)\right]$$

$$-\frac{L}{\pi}\Sigma\frac{1}{i}\left[\mathfrak{F}_i''\cosh\frac{i\pi y}{L}+\mathfrak{F}_i\cosh\frac{i\pi(D-y)}{L}\right]\sin\frac{i\pi x}{L}\Big/\sinh\frac{i\pi D}{L}$$

$$+\frac{D}{\pi}\Sigma\frac{1}{i}\left[\mathfrak{G}_i'''\sinh\frac{i\pi x}{D}-\mathfrak{G}_i'\sinh\frac{i\pi(L-x)}{D}\right]\cos\frac{i\pi y}{D}\Big/\sinh\frac{i\pi L}{D},$$

$$\frac{d\zeta}{dy}=B+Cy-xU(0,0)-\frac{xy}{D}\left[aD+U(0,D)-U(0,0)\right]$$

$$+\frac{y^2-x^2}{2L}\left[(\beta+g\rho)\,L+U(L,0)-U(0,0)\right]$$

$$+\frac{y^3-3x^2y}{6LD}\left[U(L,D)+U(0,0)-U(0,D)-U(L,0)\right]$$

$$+\frac{L}{\pi}\Sigma\frac{1}{i}\left[\mathfrak{F}_i''\sinh\frac{i\pi y}{L}-\mathfrak{F}_i\sinh\frac{i\pi(D-y)}{L}\right]\cos\frac{i\pi x}{L}\Big/\sinh\frac{i\pi D}{L}$$

$$-\frac{D}{\pi}\Sigma\frac{1}{i}\left[\mathfrak{G}_i'''\cosh\frac{i\pi x}{D}+\mathfrak{G}_i'\cosh\frac{i\pi(L-x)}{D}\right]\sin\frac{i\pi y}{D}\Big/\sinh\frac{i\pi L}{D},$$

where A and B are arbitrary constants which may be so determined as to fix any required point of the beam.

The second integration gives

$$\begin{aligned}\zeta = {} & Ax+By+\tfrac{1}{2}C\,(y^2-x^2)-xyU\,(0,\,0)\\ & +\frac{x^3-3xy^2}{6D}\left[\alpha D+U\,(0,D)-U\,(0,\,0)\right]\\ & +\frac{y^3-3x^2y}{6L}\left[(\beta+g\rho)\,L+U\,(L,\,0)-U\,(0,\,0)\right]\\ & +\frac{x^4-6x^2y^2+y^4}{24LD}\left[U(L,D)-U(L,0)-U(0,D)+U\,(0,0)\right]\\ & +\frac{L^2}{\pi^2}\sum_{i=1}^{i=\infty}\frac{1}{i^2}\left[\mathfrak{F}_i''\cosh\frac{i\pi y}{L}+\mathfrak{F}_i\cosh\frac{i\pi\,(D-y)}{L}\right]\cos\frac{i\pi x}{L}\Big/\sinh\frac{i\pi D}{L}\\ & +\frac{D^2}{\pi^2}\sum_{i=1}^{i=\infty}\frac{1}{i^2}\left[\mathfrak{G}_i''\cosh\frac{i\pi x}{D}+\mathfrak{G}_i\cosh\frac{i\pi\,(L-x)}{D}\right]\cos\frac{i\pi y}{D}\Big/\sinh\frac{i\pi L}{D}\end{aligned}$$

...............(25).

It only remains to determine the four constants α, β, γ, C. This is easily done by substituting in the equation $P+Q=2\alpha x+2\beta y+2\gamma$ the coordinates (0, 0), (L, 0), (0, D) successively, and in the equation $P-V=(2\beta+g\rho)\,y+\gamma$ the coordinates (0, 0). We thus find

$$\left.\begin{aligned}\alpha &= \frac{1}{2L}\left[\mathfrak{f}_0''+\mathfrak{f}_0'+\Sigma\,(\mathfrak{f}_i''+\mathfrak{f}_i'+2\mathfrak{g}_{2i-1})\right]\\ \beta &= \frac{1}{2D}\left[\mathfrak{g}_0''+\mathfrak{g}_0+\Sigma\,(\mathfrak{g}_i''+\mathfrak{g}_i+2\mathfrak{f}_{2i-1}')\right]\\ \gamma &= -\tfrac{1}{2}\left[\mathfrak{g}_0+\mathfrak{f}_0'+\Sigma\,(\mathfrak{g}_i+\mathfrak{f}_i')\right]\\ C &= \tfrac{1}{2}\Big\{\mathfrak{g}_0-\mathfrak{f}_0'+\Sigma\Big[\mathfrak{g}_i-\mathfrak{f}_i'-2\left(\mathfrak{F}_i''+\mathfrak{F}_i\cosh\frac{i\pi D}{L}\right)\Big/\sinh\frac{i\pi D}{L}\\ &\qquad +2\left(\mathfrak{G}_i''+\mathfrak{G}_i\cosh\frac{i\pi L}{D}\right)\Big/\sinh\frac{i\pi L}{D}\Big]\Big\}\end{aligned}\right\}\ldots(26).$$

Equations (25) and (26) present the complete solution for all cases of equilibrium in which the stress is entirely plane, and in magnitude independent of z. In order that this may be the case, certain identical relations must exist between the components of the surface stresses. These relations will of course be identical with the conditions that the form (24) of V, after substitution of the value of C given by (26), may give to the normal

stress components their assigned values over the bounding surfaces: *e.g.*, when $y = 0$,

$$C-\gamma-\mathfrak{g}_0-\frac{x}{D}\left[2aD+U(0,D)-U(0,0)\right]$$

$$-\frac{x^2}{2LD}\left[U(L,D)+U(0,0)-U(0,D)-U(L,0)\right]$$

$$-\Sigma\left[\mathfrak{G}_i'''\cosh\frac{i\pi x}{D}+\mathfrak{G}_i'\cosh\frac{i\pi(L-x)}{D}\right]\Big/\sinh\frac{i\pi L}{D}$$

$$=\Sigma\left\{\mathfrak{g}_i-\left[\mathfrak{F}_i''+\mathfrak{F}_i\cosh\frac{i\pi D}{L}\right]\Big/\sinh\frac{i\pi D}{L}\right\}\cos\frac{i\pi x}{L}.$$

Throwing the other three equations into a similar form, expanding the functions on the left-hand side by formula (21), and equating constant terms and coefficients of $\cos i\pi x/L$ or $\cos i\pi y/D$ in each, we have twelve equations of condition to be satisfied identically by the 24 coefficients (of orders 0, $2i$, and $2i-1$) in the expressions (22) for the surface tractions. The fact that the coefficients of shearing-stress appear in these equations *in series* makes the inverse problem—of determining distributions of surface traction competent to maintain a plane stress uniformly distributed in parallel planes—a difficult one, and I have not yet succeeded in obtaining consistent solutions for the cases of a beam supported by one end only, or symmetrically by both ends, with its upper and under surfaces free from stress, or uniformly loaded. I am inclined to believe that the stress in these cases is not of the kind that we have discussed, but the question requires further consideration.*

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* I am glad to express my indebtedness to Professor Greenhill for the reference to Clerk Maxwell's previous work in this direction, and to Professor J. J. Thomson for suggestions in connection with additions that have been made to this paper since it was read before the Society.

Electrical Oscillations on Cylindrical Conductors.

By Prof. J. J. THOMSON, M.A., F.R.S.

[*Read June* 10*th*, 1886.]

In a paper read before the Mathematical Society in April, 1884, I considered the electrical oscillations which can take place on spherical conductors; in this paper I propose to consider the corresponding problem when the conductors are cylindrical, as well as some other problems which can be solved by the same mathematical analysis.

The transmission of electrical waves along an infinitely long circular cylinder has been investigated by Kirchhoff (*Collected Works*, p. 131) and v. Helmholtz (*Collected Works*, Vol. I., p. 603); but, as in these investigations the effect of what Maxwell calls the displacement currents in the dielectric is not taken into account, I have thought it might be of interest to give a solution of the same problem taking the dielectric currents into account.

The first case I shall consider is that of a long circular cylinder made of a substance which conducts electricity, and entirely surrounded by a dielectric medium.

Let the axis of the cylinder be taken as the axis of z, and let F, G, H be the components of the vector potential parallel to the axes of x, y, z respectively; let ϕ be the electrostatic potential, σ the specific resistance of the substance of which the cylinder is made, μ and μ' the coefficients of magnetic permeability of the conductor and dielectric respectively, K the specific inductive capacity of the dielectric.

Let us suppose that all the variable quantities vary as ϵ^{ipt}. Then in the conductor the equations are

$$\left.\begin{aligned}\frac{\sigma}{4\pi\mu}\nabla^2 F &= \frac{dF}{dt}+\frac{d\phi}{dx}\\ \frac{\sigma}{4\pi\mu}\nabla^2 G &= \frac{dG}{dt}+\frac{d\phi}{dy}\\ \frac{\sigma}{4\pi\mu}\nabla^2 H &= \frac{dH}{dt}+\frac{d\phi}{dz}\end{aligned}\right\} \quad \text{..........................(1).}$$

In the dielectric the equations are

$$\frac{1}{\mu' K}\nabla^2 F = \frac{d^2F}{dt^2}+\frac{d^2\phi}{dt\,dx} \quad \text{..........................(2),}$$

with similar equations for G and H; and both in the conductor and dielectric
$$\nabla^2\phi = 0.$$

Thus a particular integral of the equations in the conductor is

$$F = -\frac{1}{ip}\frac{d\phi}{dx},$$

$$G = -\frac{1}{ip}\frac{d\phi}{dy},$$

$$H = -\frac{1}{ip}\frac{d\phi}{dz}.$$

The complementary functions are solutions of differential equations of the type
$$\nabla^2 H - \frac{4\pi\mu}{\sigma} ipH = 0.$$

Let all the variable functions vary as ϵ^{imz}, then this equation becomes

$$\frac{d^2H}{dx^2} + \frac{d^2H}{dy^2} - \left(m^2 + \frac{4\pi\mu}{\sigma} ip\right) H = 0,$$

or, if
$$n^2 = m^2 + \frac{4\pi\mu}{\sigma} ip,$$

we get, transforming to polar coordinates ρ and θ,

$$\frac{d^2H}{d\rho^2} + \frac{1}{\rho}\frac{dH}{d\rho} + \frac{1}{\rho^2}\frac{d^2H}{d\theta^2} - n^2H = 0.$$

Or, since H is supposed not to depend upon θ,

$$\frac{d^2H}{d\rho^2} + \frac{1}{\rho}\frac{dH}{d\rho} - n^2H = 0.$$

Since the solution of this equation is finite when ρ is zero, it must be a multiple of $J_0(in\rho)$, where J_0 denotes Bessel's function of zero order; thus, taking the particular integral into account, the value of H inside the cylinder is given by the equation

$$H = BJ_0(in\rho) - \frac{1}{ip}\frac{d\phi}{dz}.$$

Since $\phi \propto \epsilon^{imz}$, and is independent of the angular coordinate θ, the equation
$$\nabla^2\phi = 0$$
transforms to
$$\frac{d^2\phi}{d\rho^2} + \frac{1}{\rho}\frac{d\phi}{d\rho} - m^2\phi = 0,$$

the solution of which, when ρ can vanish, is

$$\phi = AJ_0(im\rho).$$

When ρ can become infinite, the solution is

$$\phi = A'I_0(im\rho)\,\epsilon^{imz},$$

where $I_0(im\rho)$ denotes a Bessel's function of zero order of the second kind. Inside the cylinder

$$H = \left\{ BJ_0(in\rho) - \frac{im}{ip}AJ_0(im\rho) \right\} \epsilon^{imz} \quad\text{..............(3).}$$

In order to find the complementary function for F and G, we notice that we can by symmetry put

$$F = \frac{d\chi}{dx}, \qquad G = \frac{d\chi}{dy},$$

where χ satisfies the equation

$$\frac{d^2\chi}{d\rho^2} + \frac{1}{\rho}\frac{d\chi}{d\rho} - n^2\chi = 0,$$

so that inside the cylinder

$$\chi = CJ_0(in\rho)\,\epsilon^{imz},$$

and therefore

$$\left.\begin{aligned} F &= \frac{d}{dx}\left\{ CJ_0(in\rho) - \frac{1}{ip}AJ_0(im\rho) \right\} \epsilon^{imz} \\ G &= \frac{d}{dy}\left\{ CJ_0(in\rho) - \frac{1}{ip}AJ_0(im\rho) \right\} \epsilon^{imz} \end{aligned}\right\} \quad\text{.........(4).}$$

Since $\dfrac{dF}{dx} + \dfrac{dG}{dy} + \dfrac{dH}{dz} = 0$, we have

$$n^2C + im\,B = 0,$$

$$C = -\frac{im}{n^2}B.$$

In the dielectric, we can easily prove, in a similar way, that

$$\left.\begin{aligned} H &= \left\{ DI_0(i\kappa\rho) - \frac{im}{ip}A'I_0(im\rho) \right\} \epsilon^{imz} \\ F &= \frac{d}{dx}\left\{ EI_0(i\kappa\rho) - \frac{1}{ip}A'I_0(im\rho) \right\} \epsilon^{imz} \\ G &= \frac{d}{dy}\left\{ EI_0(i\kappa\rho) - \frac{1}{ip}A'I_0(im\rho) \right\} \epsilon^{imz} \end{aligned}\right\} \quad\text{...........(5),}$$

where $$\kappa^2 = m^2 - p^2\mu' K,$$

and $$\kappa^2 E + im D = 0,$$

$$E = -\frac{im}{\kappa^2} D.$$

Thus we have the following equations :—

Inside the cylinder,

$$\phi = AJ_0(im\rho)\, \epsilon^{imz}\, \epsilon^{ipt},$$

$$H = \left\{ BJ_0(in\rho) - \frac{im}{ip} AJ_0(im\rho) \right\} \epsilon^{imz}\, \epsilon^{ipt},$$

$$F = \frac{d\chi'}{dx}, \qquad G = \frac{d\chi'}{dy},$$

$$\chi' = \left\{ -\frac{im}{n^2} BJ_0(in\rho) - \frac{1}{ip} AJ_0(im\rho) \right\} \epsilon^{imz}\, \epsilon^{ipt}.$$

In the dielectric

$$\phi = A'I_0(im\rho)\, \epsilon^{imz}\, \epsilon^{ipt},$$

$$H = \left\{ DI_0(i\kappa\rho) - \frac{im}{ip} A'I_0(im\rho) \right\} \epsilon^{imz}\, \epsilon^{ipt},$$

$$F = \frac{d\chi'}{dx}, \qquad G = \frac{d\chi'}{dy},$$

$$\chi' = \left\{ -\frac{im}{\kappa^2} DI_0(i\kappa\rho) - \frac{1}{ip} A'I_0(im\rho) \right\} \epsilon^{imz}\, \epsilon^{ipt}.$$

Now ϕ is continuous as we cross from the conductor to the dielectric, and F, G, H and their differential coefficients are also continuous.

If a be the radius of the cylinder, we have, since ϕ is continuous,

$$AJ_0(ima) = A'I_0(ima);$$

since H is continuous, $BJ_0(ina) = DI_0(i\kappa a)$;

since F and G are continuous,

$$\frac{m}{n} BJ_0'(ina) - \frac{m}{p} AJ_0'(ima) = \frac{m}{\kappa} DI_0'(i\kappa a) - \frac{m}{p} A'I_0'(ima);$$

and, since $dH/d\rho$ is continuous,

$$inBJ_0'(ina) + \frac{m^2}{ip} AJ_0'(ima) = i\kappa DI_0'(i\kappa a) + \frac{m^2}{ip} A'I_0'(ima).$$

Eliminating D and A', we have

$$B\left(\frac{m}{n}J_0'(ina)-\frac{m}{\kappa}\frac{I_0'(i\kappa a)}{I_0(i\kappa a)}J_0(ina)\right)$$
$$=A\left(\frac{m}{p}J_0'(ima)-\frac{m}{p}\frac{I_0'(ima)}{I_0(ima)}J_0(ima)\right),$$
$$B\left(nJ_0'(ina)-\kappa\frac{I_0'(i\kappa a)}{I_0(i\kappa a)}J_0(ina)\right)$$
$$=A\left(\frac{m^2}{p}J_0'(ima)-\frac{m^2}{p}\frac{I_0'(ima)}{I_0(ima)}J_0(ima)\right);$$

eliminating A and B, we have

$$\frac{m^2-n^2}{n}\frac{J_0'(ina)}{J_0(ina)}=\frac{m^2-\kappa^2}{\kappa}\frac{I_0'(i\kappa a)}{I_0(i\kappa a)} \quad\text{..................(6)}.$$

Now $$m^2-\kappa^2=\frac{p^2}{V^2},$$

where V is the velocity of light in the dielectric, and

$$m^2-n^2=-\frac{4\pi\mu}{\sigma}ip,$$

so that in general m^2-n^2 is large compared with $m^2-\kappa^2$, and therefore $I_0'(i\kappa a)/I_0(i\kappa a)$ must be large, that is, κ must be small; but,

when $i\kappa a$ is small, $$I_0(i\kappa a)=-\log\gamma i\kappa a,$$

where $$\log\gamma=\cdot 577-\log 2,$$

and $$I_0'(i\kappa a)=-\frac{1}{i\kappa a},$$

so that our equation becomes

$$(i\kappa a)^2\log(\gamma i\kappa a)=\frac{ip\sigma}{4\pi\mu V^2}ina\frac{J_0(ina)}{J_0'(ina)} \quad\text{...............(7)}.$$

Let us consider the two extreme cases in one of which ina is very small, and in the other very large.

When ina is very small,

$$J_0(ina)=1,\quad J_0'(ina)=-\frac{ina}{2}.$$

So that, if z be written for $(\gamma i\kappa a)^2$,

$$z\log z=-\frac{ip\sigma}{\pi\mu V^2}\gamma^2.$$

To solve this equation, let $z=R\epsilon^{i\phi}$,

and write N for $-p\sigma\gamma^2/\pi\mu V^2$;

then, equating real and imaginary parts, we have

$$R \log R \cos\phi - R\phi \sin\phi = 0,$$

$$R \log R \sin\phi + R\phi \cos\phi = N.$$

Since R is very small, an approximate solution will be

$$R \log R = N,$$

$$\phi = \frac{\pi}{2} - \frac{R}{N}\,\frac{\pi}{2}.$$

The easiest way of solving the equation $R \log R = N$, is to make a table of $x \log x$ for various small values of x; an approximate solution, when N is very small, is, however, given by

$$R = -\frac{N}{\log N}.$$

Taking this solution, we have

$$\kappa^2 = -\frac{p\sigma}{\pi\mu V^2 a^2 \log p\sigma\gamma^2/\pi\mu V^2}\left\{\frac{\pi}{2}\,\frac{1}{\log p\sigma\gamma^2/\pi\mu V^2} + i\right\},$$

therefore

$$m^2 = \frac{p^2}{V^2} - \frac{p\sigma}{\pi\mu V^2 a^2 \log p\sigma\gamma^2/\pi\mu V^2}\left\{\frac{\pi}{2}\,\frac{1}{\log p\sigma\gamma^2/\pi\mu V^2} + i\right\};$$

so that, if $\sigma/\pi\mu p a^2 \log p\sigma\gamma^2/\pi\mu V^2$ be small, we have

$$m = \frac{p}{V}\left\{1 - \frac{1}{2}\,\frac{i\sigma}{p\pi\mu a^2 \log p\sigma\gamma^2/\pi\mu V^2}\right\} \quad \ldots\ldots\ldots\ldots\ldots(8).$$

This represents a vibration propagated with the velocity of light, the amplitude of which fades away to $1/e$ of its original value after the wave has travelled over a length $-2\pi\mu\sigma^{-1}a^2 V \log p\sigma\gamma^2/\pi\mu V^2$; this distance is practically the same for vibrations of all periods, so that a disturbance will not lose its characteristic features as it travels along the cylinder.

The condition for this, as we have just seen, is that

$$\frac{\sigma}{p\pi\mu a^2 \log p\sigma\gamma^2/\pi\mu V^2},$$

or the approximately equal expression

$$\frac{4}{n^2 a^2 \log p\sigma\gamma^2/\pi\mu V^2},$$

should be small. Our solution, however, is founded on the assumption that na is small, so that these assumptions are to a certain extent antagonistic; they may, however, be reconciled by making n^2a^2 moderately small, say between $\frac{1}{5}$ and $\frac{1}{2}$.

As a numerical illustration of this, let us take the case of a copper wire for which $\sigma = 1500$. By the use of the formula given above, we find that if the wire be about $\frac{1}{8}$ of a centimetre in radius, all vibrations between 100 and 500 a second will fade away equally rapidly, and the distance the disturbance will travel before its amplitude falls to $1/e$ of its original value is about 1000 miles.*

Let us now consider the case when na is large; since in this case $J_0'(ina) = -iJ_0(ina)$, equation (7), gives, writing z for $(\gamma i\kappa a)^2$,

$$\left.\begin{aligned} z\log z &= -\frac{ip\sigma\gamma^2}{4\pi\mu V^2}na \\ &= \frac{p\sigma a\gamma^2}{4\pi\mu V^2}\sqrt{\frac{2\pi\mu p}{\sigma}}(1-i) \\ &= M(1-i) \text{ say} \end{aligned}\right\} \qquad (9).$$

Let
$$z = R\epsilon^{i\phi}.$$

Then, equating real and imaginary parts,

$$R\log R\cos\phi - R\phi\sin\phi = M,$$
$$R\log R\sin\phi + R\phi\cos\phi = -M.$$

Since $\log R$ is very large, we may satisfy these equations approximately by the assumption

$$R\log R = M\sqrt{2},$$
$$\phi = -\frac{\pi}{4};$$

therefore, approximately, $$R = \frac{-M\sqrt{2}}{\log M\sqrt{2}},$$

therefore $$\kappa^2 a^2 = \frac{1}{\gamma^2}\,\frac{M\sqrt{2}}{\log M\sqrt{2}}\left\{\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right\},$$

therefore $$m^2 = \frac{p^2}{V^2} + \frac{1}{a^2\gamma^2}\,\frac{M}{\log M\sqrt{2}}\{1-i\};$$

so that, as $\dfrac{MV^2}{p^2a^2\gamma^2\log M\sqrt{2}}$ is small, we may write

$$m = \pm\frac{p}{V}\left\{1 - \tfrac{1}{2}i\,\frac{MV^2}{p^2a^2\gamma^2\log M\sqrt{2}}\right\} \qquad (10).$$

---

* [When na is small and $\dfrac{4}{n^2a^2\log p\sigma\gamma^2/\pi\mu V^2}$ large, the case coincides with that discussed later on in equation (24), writing $\log p\sigma\gamma^2/\pi\mu V^2$ for the $\log b/a$ which occurs in that equation. The velocity of propagation is much smaller in this case.]

This, as before, represents a vibration propagated with the velocity of light, and sinking to $1/e$ of its original value after traversing a space $\frac{2p}{V}\frac{a^2\gamma^2 \log M\sqrt{2}}{M}$; on substituting for M its value, this becomes

$$8Va\left\{\frac{\pi\mu}{2\sigma p}\right\}^{\frac{1}{2}} \log \frac{ap^{\frac{1}{2}}\gamma^2}{V^2}\left\{\frac{\sigma}{4\pi\mu}\right\}^{\frac{1}{2}}.$$

This differs very materially from the case when na is small. In the first place, vibrations of different pitch no longer fade away at the same rate, the quick vibrations die away faster than the slow; secondly, the distance vibrations travel varies as $a\sigma^{-\frac{1}{2}}$ instead of $a^2\sigma^{-1}$, so that the conductivity is not so important as when na is small. As a numerical example, let us take the case of an iron wire $\frac{1}{6}$ of a centimetre in radius, for which $\sigma = 10^4$, $\mu = 500$, and let the vibration be one in which there are 100 vibrations a second; then the vibration will travel about 100,000 miles before sinking to $1/e$ of its original value, so that for vibrations of this speed the iron wire would carry further than the copper one; for much quicker vibrations, the copper one would be the better. When na is smaller, $J_0(in\rho)$ is nearly unity, so that the currents are uniformly distributed throughout the cylinder; when ina is large, $J_0(in\rho)$ is very large near the boundary, but equal to unity in the middle of the wire, in this case the currents are chiefly near the surface of the wire; so that we can understand why the distance the vibration travels varies as the circumference, and not the area of the cross section of the wire.

Let us now take the case of a cylindrical conductor separated by a coaxal cylindrical insulating sheath from an infinite mass of another conductor—the case, that is, of a submarine cable.

Let μ, σ be respectively the magnetic permeability and specific resistance of the core;

μ', K the magnetic permeability and specific inductive capacity of the sheath;

μ_1, σ_1 the magnetic permeability and specific resistance of the outside conductor.

Let us assume, as before, that all the variable quantities vary as $e^{i(mz+pt)}$. Then, if

$$n^2 = m^2 + \frac{4\pi\mu ip}{\sigma},$$

$$n'^2 = m^2 + \frac{4\pi\mu' ip}{\sigma'},$$

$$\kappa^2 = m^2 - \frac{p^2}{V^2},$$

where V is the velocity of light in the sheath, we may easily prove, using the same notation as before, that

$$\begin{aligned}\phi &= A\epsilon^{imz}\epsilon^{ipt}J_0(im\rho), \text{ in the core,}\\ &= \epsilon^{imz}\epsilon^{ipt}\{BJ_0(im\rho)+CI_0(im\rho)\}, \text{ in the sheath,}\\ &= D\epsilon^{imz}\epsilon^{ipt}I_0(im\rho), \text{ in the outside conductor,}\end{aligned}$$

$$\begin{aligned}\frac{1}{ip}\frac{d\phi}{dz}+H &= \epsilon^{imz}\epsilon^{ipt}EJ_0(in\rho), \text{ in the core,}\\ &= \epsilon^{imz}\epsilon^{ipt}\{FJ_0(i\kappa\rho)+GI_0(i\kappa\rho)\}, \text{ in the sheath,}\\ &= \epsilon^{imz}\epsilon^{pt}HI_0(in'\rho), \text{ in the outside conductor,}\end{aligned}$$

$$F=\frac{d\chi}{dx}, \qquad G=\frac{d\chi}{dy},$$

where
$$\begin{aligned}\chi &= -\epsilon^{imz}\epsilon^{ipt}\frac{im}{n^2}EJ_0(in\rho)-\frac{1}{ip}\phi, \text{ in the core,}\\ &= -\epsilon^{imz}\epsilon^{ipt}\frac{im}{\kappa^2}\{FJ_0(i\kappa\rho)+GI_0(i\kappa\rho)\}-\frac{1}{ip}\phi, \text{ in the sheath,}\\ &= -\epsilon^{imz}\epsilon^{ipt}\frac{im}{n_1^2}HI_0(in'\rho)-\frac{1}{ip}\phi, \text{ in the outside conductor.}\end{aligned}$$

If a and b be the internal and external diameters of the sheath, the continuity of ϕ requires that

$$\left.\begin{aligned}AJ_0(ima) &= BJ_0(ima)+CI_0(ima)\\ DI_0(imb) &= BJ_0(imb)+CI_0(imb)\end{aligned}\right\}\text{..............}(11);$$

the continuity of H requires that

$$\left.\begin{aligned}EJ_0(ina) &= FJ_0(i\kappa a)+GI_0(i\kappa a)\\ HI_0(in'b) &= FJ_0(i\kappa b)+GI_0(i\kappa b)\end{aligned}\right\}\text{..............}(12);$$

the continuity of F and G requires that

$$\begin{aligned}-\frac{im}{n^2}inJ_0'(ina)E-\frac{1}{ip}imJ_0'(ima)A\\ = -\frac{im}{\kappa^2}i\kappa\{FJ_0'(i\kappa a)+GI_0'(i\kappa a)\}\\ -\frac{im}{ip}\{BJ_0'(ima)+CI_0'(ima)\}\text{ ....}(13),\end{aligned}$$

$$-\frac{im}{n'^2}\, in' I_0'(inb)\, H - \frac{1}{ip}\, im\, I_0'(imb)\, D$$

$$= -\frac{im}{\kappa^2}\, i\kappa\, (FJ_0'(i\kappa b) + GI_0'(i\kappa b)\}$$

$$-\frac{im}{ip}\{BJ_0'(imb) + CI_0'(imb)\} \;\ldots\ldots\ldots\ldots\ldots(14);$$

the continuity of $dH/d\rho$ requires that

$$in J_0'(ina)\, E + \frac{m^2}{ip} J_0'(ima)\, A$$

$$= i\kappa\,\{FJ_0'(i\kappa a) + GI_0'(i\kappa a)\} + \frac{m^2}{ip}\{BJ_0'(ima) + CI_0'(ima)\}\ldots(15),$$

$$in' I_0'(in'b)\, H + \frac{m^2}{p} I_0'(imb)\, D$$

$$= i\kappa\,\{FJ_0'(i\kappa b) + GI_0'(i\kappa b)\} + \frac{m^2}{ip}\{BJ_0'(imb) + CI_0'(imb)\}\;\ldots(16).$$

Eliminating A, D, E, and H from these equations, we find

$$F\left\{\frac{m}{\kappa} J_0'(i\kappa a) - \frac{m}{n}\frac{J_0(i\kappa a)}{J_0(ina)} J_0'(ina)\right\}$$

$$+ G\left\{\frac{m}{\kappa} I_0'(i\kappa a) - \frac{m}{n}\frac{I_0(i\kappa a)}{J_0(ina)} J_0'(ina)\right\}$$

$$= \frac{2m}{p} CI_0'(ima) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

$$F\left\{\kappa J_0'(i\kappa a) - n\frac{J_0(i\kappa a)}{J_0(ina)} J_0'(ina)\right\}$$

$$+ G\left\{\kappa I_0'(i\kappa a) - n\frac{I_0(i\kappa a)}{J_0(ina)} J_0'(ina)\right\}$$

$$= \frac{2m^2}{p} CI_0'(ima) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18);$$

so that $F\left\{\dfrac{m^2-\kappa^2}{\kappa} J_0'(i\kappa a)\, J_0(ina) - \dfrac{m^2-n^2}{n} J_0(i\kappa a)\, J_0'(ina)\right\}$

$$+ G\left\{\frac{m^2-\kappa^2}{\kappa} I_0'(i\kappa a)\, J_0(ina) - \frac{(m^2-n^2)}{n} I_0(i\kappa a)\, J_0'(ina)\right\} = 0$$

$$\ldots\ldots\ldots\ldots\ldots(19).$$

Similarly, we find that

$$F\left\{\frac{m^2-\kappa^2}{\kappa}J_0'(i\kappa b)\,I_0(in'b)-\frac{m^2-n'^2}{n'}J_0(i\kappa b)\,I_0'(in'b)\right\}$$

$$+G\left\{\frac{m^2-\kappa}{\kappa}I_0'(i\kappa b)\,I_0(in'b)-\frac{m^2-n^2}{n}I_0(i\kappa b)\,I_0'(in'b)\right\}=0 \ldots (20),$$

so that

$$\left\{\frac{m^2-\kappa^2}{\kappa}J_0'(i\kappa a)\,J_0(in a)-\frac{m^2-n^2}{n}J_0(i\kappa a)\,J_0'(in a)\right\}$$

$$\times\left\{\frac{m^2-\kappa^2}{\kappa}I_0'(i\kappa b)\,I_0(in'b)-\frac{m^2-n'^2}{n'}I_0(i\kappa b)\,I_0'(in'b)\right\}$$

$$=\left\{\frac{m^2-\kappa^2}{\kappa}I_0'(i\kappa a)\,J_0(in a)-\frac{m^2-n^2}{n}I_0(i\kappa a)\,J_0'(in a)\right\}$$

$$\times\left\{\frac{m^2-\kappa^2}{\kappa}J_0'(i\kappa b)\,I_0(in'b)-\frac{m^2-n^2}{n}J_0(i\kappa b)\,I_0'(in'b)\right\} \ldots (21).$$

This equation gives the relation between the wave-length and the time of vibration for any electrical oscillation. We can simplify it very much by remembering that κa and κb are very small. Making this supposition, and introducing the approximate values given before for $I_0(i\kappa a)$, $I_0'(i\kappa a)$, when κa is very small, we find

$$\kappa^2=\frac{ip^2}{V^2}\left\{\frac{n}{m^2-n^2}\,\frac{1}{a}\,\frac{J_0(in a)}{J_0'(in a)}-\frac{n'}{m^2-n'^2}\,\frac{1}{b}\,\frac{I_0(in'b)}{I_0'(in'b)}\right\}\frac{1}{\log b/a}\ldots (22).$$

Now, in practice, the outer conductor will be water, for which $\sigma=$ about 5×10^{10} and $\mu=1$; so that, except for very rapid oscillations, $n'b$ will be very small; but in this case $I_0(in'b)/I_0'(in'b)$ is very small, so that we may write

$$\kappa^2=\frac{ip^2}{V^2}\,\frac{n}{m^2-n^2}\,\frac{1}{a}\,\frac{J_0(in a)}{J_0'(in a)}\,\frac{1}{\log b/a}\ \ldots\ldots\ldots\ldots\ldots (23).$$

Let us first consider the case when na is very small; in this case

$$\kappa^2=\frac{2p^2}{V^2a^2}\,\frac{\sigma}{4\pi\mu ip}\,\frac{1}{\log b/a},$$

$$m^2=\frac{p^2}{V^2}\left\{1-\frac{i\sigma}{p\,a^2\,2\mu\pi\log b/a}\right\}\ \ldots\ldots\ldots\ldots\ldots (24).$$

If na be small, the second term will be large compared with the first,

since we cannot make $\log b/a$ very large, so that in this case

$$m^2 = -\frac{i\sigma p^2}{V^2a^2 2\pi\mu p}\,\frac{1}{\log b/a}, \text{ approximately,}$$

$$m = \left\{\frac{p\sigma}{V^2a^2 4\pi\mu \log b/a}\right\}^{\frac{1}{2}}(1-i).$$

This represents a wave travelling with the velocity

$$V\left\{\frac{4\pi p\mu\, a^2 \log b/a}{\sigma}\right\}^{\frac{1}{2}},$$

the amplitude falling to $1/e$ of its value after the wave has passed over a distance

$$V\left\{\frac{4\pi\mu\, a^2 \log b/a}{p\sigma}\right\}^{\frac{1}{2}}.$$

From the above formulæ we see that the velocity of propagation of the wave in knots per second

$$=\left\{\frac{2p}{\text{resistance of a knot of the cable} \times \text{capacity of a knot}}\right\}^{\frac{1}{2}}.$$

In this case the wave travels much more slowly, and fades away much more quickly, than in the case we discussed before, when the wire was not surrounded by another conductor.

Let us take the case of a copper cable conveying an oscillation whose period is $\frac{1}{100}$ of a second. Suppose the resistance of the cable is 5 ohms per knot, and the electrostatic capacity ·3 of a microfarad per knot. Then an electrical wave would be propagated along the cable at the rate of about 30,000 miles per second, and the amplitude of the vibration would fall to $1/e$ of its original value, after passing over about 50 miles, so that it would not be possible to convey telephonic messages over much more than 100 miles of such a cable.

Let us next take the case when na is large. Since in this case $J_0'(ina) = -inJ_0(ina)$, equation (23) gives approximately

$$\kappa^2 = \frac{p^2}{V^2 na \log b/a},$$

$$m^2 = \frac{p^2}{V^2} + \frac{p^2}{V^2 na \log b/a} \quad \text{..................... (25).}$$

Since na is large, the second term is small compared with the first, so that in this case the wave is propagated with the velocity of light. To find the rate at which the vibrations die away, we must remember that approximately, since m is small,

$$n^2 = \frac{4\pi\mu ip}{\sigma},$$

and therefore $$\frac{1}{na} = \left\{\frac{\sigma}{8\pi\mu pa^2}\right\}^{\frac{1}{2}}(1-i),$$

hence $$m = \frac{p}{V}\left[1 - \frac{i}{2}\left\{\frac{\sigma}{8\pi\mu pa^2}\right\}^{\frac{1}{2}}\frac{1}{\log b/a}\right] \text{ approximately } \ldots(26).$$

So that the distance the wave must travel before its amplitude falls to $1/e$ of its original value is

$$2\,Va \log b/a \left\{\frac{8\pi\mu}{\sigma p}\right\}^{\frac{1}{2}}.$$

Let us take the case of an iron wire for which $\mu = 500$, $\sigma = 10^4$; let the period of the oscillation be $\frac{1}{100}$ of a second, then

$$n^2 = 4\pi\mu ip/\sigma = 400i,$$

so that, if the wire be more than $\frac{1}{2}$ a centimetre in radius, na will be large, and the distance the oscillation will travel before its amplitude is reduced to $1/e$ of its original value is $2a \log b/a \times 13 \times 10^8$. Suppose $2a = 1$ and $\log\ b/a = \frac{1}{4}$, then the distance would be over 2000 miles, so the vibrations would die away much more slowly than for the copper wire.

Thus, if this theory be correct, if equal periodic disturbances are started in an iron-wire and a copper-wire cable respectively, they will travel a much greater distance along the iron than along the copper.

Let us next investigate the importance of any leakage which there may be across the insulating sheath. If the specific resistance of the sheath be s, then the equations satisfied by the vector potential in the sheath are of the form

$$\mu\left\{\frac{4\pi}{s} + K\frac{d}{dt}\right\}\left\{\frac{dH}{dt} + \frac{d\phi}{dz}\right\} = \nabla^2 H$$

(Maxwell's *Electricity and Magnetism*, 2nd Edition, Vol. II., p. 395), or, since H varies as ϵ^{ipt},

$$\left\{\frac{4\pi\mu\, ip}{s} - \mu\kappa p^2\right\}H = \nabla^2 H - \left\{\frac{4\pi\mu}{s} + i\kappa p\right\}\frac{d\phi}{dz};$$

so that, if we put $$\kappa_1^2 = m^2 - \frac{p^2}{V^2} + \frac{4\pi\mu\, ip}{s},$$

the solution will be the same as before, with κ_1 written instead of κ. So that we shall have the same equation as before, with $p^2/V^2 - 4\pi\mu\, ip/s$ instead of p^2/V^2. Thus the leakage will, or will not, be important according as $\dfrac{4\pi\mu\, ip/s}{p^2/V^2}$ is, or is not, large. Now, for gutta-percha s is

about $4{\cdot}5 \times 10^{23}$ (Everett's *Physical Constants*, p. 145); so that, if the period of the oscillation be $\frac{1}{100}$ of a second,

$$\frac{4\pi\mu ip/s}{p^2/V^2} = 2 \times 10^{-5} \text{ approximately};$$

so that, if the insulating sheath be made of gutta-percha, or any better insulator, the leakage is quite unimportant.

In many important cases in practice, a current flowing along a wire is accompanied by a return current flowing along a wire by the side of it. In these cases there is a want of symmetry, which would make the mathematical solution of it very difficult. If, however, the wires conveying the direct and the return currents are near together, the case may be expected to present much the same physical features as the case of a plane slab of dielectric separating two infinite conductors separated by parallel plane faces; and this case, as the following investigation will show, admits of a simple solution.

Let the normal to the face of the slab be taken as the axis of x, and let the origin be half-way between the faces of the slab, whose thickness we shall suppose to be $2h$.

Let μ and σ be respectively the magnetic permeability and specific resistance of the conductors;

μ' and K the magnetic permeability and specific inductive capacity of the dielectric.

Let us suppose, as before, that all the variable quantities vary as ϵ^{imz}, ϵ^{ipt}.

Then we may easily prove that, using the same notation as before,

$$\begin{aligned}\phi &= \epsilon^{imz}\epsilon^{ipt}\left\{A\epsilon^{mx}+B\epsilon^{-mx}\right\} \text{ in the dielectric,}\\ &= \epsilon^{imz}\epsilon^{ipt}\,C\epsilon^{-mx} \text{ in the conductor on the positive side of the origin,}\\ &= \epsilon^{imz}\epsilon^{ipt}\,D\epsilon^{mx} \text{ in the conductor on the negative side of the origin,}\end{aligned}$$

$$\begin{aligned}H+\frac{1}{ip}\frac{d\phi}{dz} &= \epsilon^{imz}\epsilon^{ipt}\left\{E\epsilon^{\kappa x}+F\epsilon^{-\kappa x}\right\} \text{ in the dielectric,}\\ &= \epsilon^{imz}\epsilon^{ipt}\,G\epsilon^{-nx} \text{ in the conductor on the positive side of the origin,}\\ &= \epsilon^{imz}\epsilon^{ipt}\,H\epsilon^{nx} \text{ in the conductor on the negative side of the origin,}\end{aligned}$$

$$F+\frac{1}{ip}\frac{d\phi}{dx}=-\frac{im}{R}\,\epsilon^{imz}\epsilon^{ipt}\left\{E\epsilon^{\kappa x}-F\epsilon^{-\kappa x}\right\}$$ in the dielectric,

$$=\frac{im}{n}\,\epsilon^{imz}\epsilon^{ipt}\,G\epsilon^{-nx}$$ in the conductor on the positive side of the origin,

$$=-\frac{im}{n}\,H\epsilon^{nx}$$ in the conductor on the negative side of the origin,

$$G=0.$$

Then, from the continuity of ϕ, we have the following equation,

$$\left.\begin{aligned}A\epsilon^{mh}+B\epsilon^{-mh}&=C\epsilon^{-mh}\\ A\epsilon^{-mh}+B\epsilon^{mh}&=D\epsilon^{-mh}\end{aligned}\right\}\text{......................}(27);$$

from the continuity of H, we have

$$\left.\begin{aligned}E\epsilon^{\kappa h}+F\epsilon^{-\kappa h}&=G\epsilon^{-nh}\\ E\epsilon^{-\kappa h}+F\epsilon^{\kappa h}&=H\epsilon^{-nh}\end{aligned}\right\}\text{......................}(28);$$

from the continuity of F, we have

$$\left.\begin{aligned}-\frac{im}{\kappa}\left\{E\epsilon^{\kappa h}-F\epsilon^{-\kappa h}\right\}-\frac{im}{ip}\left\{A\epsilon^{mh}-B\epsilon^{-mh}\right\}&=\frac{im}{n}G\epsilon^{-nh}+\frac{m}{ip}C\epsilon^{-mh}\\ -\frac{im}{\kappa}\left\{E\epsilon^{-\kappa h}-F\epsilon^{\kappa h}\right\}-\frac{m}{ip}\left\{A\epsilon^{-mh}-B\epsilon^{mh}\right\}&=-\frac{im}{n}H\epsilon^{-nh}-\frac{m}{ip}D\epsilon^{-mh}\end{aligned}\right\}$$

$$\text{..................}(29);$$

from the continuity of dH/dx, we have

$$\left.\begin{aligned}\kappa\left\{E\epsilon^{\kappa h}-F\epsilon^{-\kappa h}\right\}-\frac{im^2}{ip}\left\{A\epsilon^{mh}-B\epsilon^{-mh}\right\}&=-nG\epsilon^{-nh}+\frac{m^2}{p}C\epsilon^{-mh}\\ K\left\{E\epsilon^{-\kappa h}-F\epsilon^{\kappa h}\right\}-\frac{m^2}{p}\left\{A\epsilon^{-mh}-B\epsilon^{mh}\right\}&=nH\epsilon^{-nh}-\frac{m^2}{p}D\epsilon^{-mh}\end{aligned}\right\}$$

$$\text{..................}(30).$$

Eliminating C, D, G, H, we see that

$$-E\epsilon^{\kappa h}\left\{\frac{1}{\kappa}+\frac{1}{n}\right\}+F\epsilon^{-\kappa h}\left\{\frac{1}{\kappa}-\frac{1}{n}\right\}=-\frac{2}{p}A\epsilon^{-mh}\text{...}(31),$$

$$E\epsilon^{-\kappa h}\left\{\frac{1}{n}-\frac{1}{\kappa}\right\}+F\epsilon^{\kappa h}\left\{\frac{1}{\kappa}+\frac{1}{n}\right\}=\frac{2}{p}B\epsilon^{mh}\text{...}(32),$$

$$E\epsilon^{\kappa h}\left\{\kappa+n\right\}+F\epsilon^{-\kappa h}\left\{n-\kappa\right\}=\frac{2m^2}{p}A\epsilon^{-mh}\text{...}(33),$$

$$E\epsilon^{-\kappa h}\left\{\kappa-n\right\}-F\epsilon^{\kappa h}\left\{n+\kappa\right\}=-\frac{2m^2}{p}B\epsilon^{mh}\text{...}(34),$$

therefore

$$E\epsilon^{\kappa h}\;\{\kappa+n\}\left\{1-\frac{m^2}{\kappa n}\right\}+F\epsilon^{-\kappa h}\;\{n-\kappa\}\left\{1+\frac{m^2}{\kappa n}\right\}=0\;\ldots(35),$$

$$E\epsilon^{-\kappa h}\{\kappa-n\}\left\{1-\frac{m^2}{\kappa n}\right\}-F\epsilon^{\kappa h}\;\;\{\kappa+n\}\left\{1-\frac{m^2}{\kappa n}\right\}=0\;\ldots(36),$$

therefore $\epsilon^{\kappa h}\{\kappa+n\}\left\{1-\frac{m^2}{\kappa n}\right\}=\pm\epsilon^{-\kappa h}\{\kappa-n\}\left\{1+\frac{m^2}{\kappa n}\right\}$,

therefore $$\frac{\epsilon^{\kappa h}\;\{\kappa+n\}}{\epsilon^{-\kappa h}\{\kappa-n\}}=\pm\left\{1+\frac{m^2}{\kappa n}\right\}\Big/\left\{1-\frac{m^2}{\kappa n}\right\}\;\ldots\ldots\ldots\ldots(37).$$

The lower sign gives the solution of the problem we are considering, where the current goes in opposite directions in the two conductors; for it makes $E=-F$, and therefore $A=-B$, $C=-D$, so that all quantities on the positive side of the origin are equal and opposite to the same quantities in the corresponding position on the negative side.

We may write equation (37) as

$$\frac{\kappa^2-m^2}{\kappa}\left\{\epsilon^{\kappa h}+\epsilon^{-\kappa h}\right\}+\frac{n^2-m^2}{n}\left\{\epsilon^{\kappa h}-\epsilon^{-\kappa h}\right\}=0,$$

but $$\kappa^2-m^2=-p^2/V^2,$$

$$n^2-m^2=4\pi\mu ip/\sigma,$$

and κh is very small, therefore we have approximately

$$\frac{p^2}{V^2\kappa}=\frac{4\pi\mu\, ip}{\sigma n}\kappa h,$$

or $$\kappa^2=\frac{p^2}{V^2}\;\frac{n\sigma}{h.4\pi\mu ip},$$

so that $$m^2=\frac{p^2}{V^2}\left\{1+\frac{n\sigma}{4\pi\mu\, hip}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38);$$

for metals $4\pi\mu ip/\sigma$ is large compared with m^2, unless p be very large, so that, approximately, $n=\{4\pi\mu ip/\sigma\}^{\frac{1}{2}}$,

$$m^2=\frac{p^2}{V^2}\left[1+\left\{\frac{\sigma}{8\pi\mu h^2p}\right\}^{\frac{1}{2}}\{1-i\}\right]\;\ldots\ldots\ldots\ldots(38^*);$$

if $\sigma / 8\pi\mu p h^2$ be small, then, approximately,

$$m = \frac{p}{V}\left[1 - \frac{1}{2}\, i \left\{ \frac{\sigma}{8\pi\mu h^2 p} \right\}^{\frac{1}{2}}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (39).$$

This represents a vibration propagated with the velocity of light, and fading away to $1/e$ of its amplitude after traversing a distance

$$2Vh \left\{ \frac{8\pi\mu}{p\sigma} \right\}^{\frac{1}{2}}.$$

If $\sigma / 8\pi\mu p h^2$ be large, then

$$m = \frac{p}{V}\left\{ \frac{\sigma}{4\pi\mu h^2 p} \right\}^{\frac{1}{4}} \left\{ \cos\frac{\pi}{8} - i \sin\frac{\pi}{8} \right\},$$

$$m = \frac{p}{V}\left\{ \frac{\sigma}{4\pi\mu h^2 p} \right\}^{\frac{1}{4}} \{ \cdot 32 - i \, . \, 92 \} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (40).$$

This represents a vibration propagated with velocity

$$3{\cdot}1\, V \{ 4\pi\mu h^2 p / \sigma \}^{\frac{1}{4}},$$

and fading away to $1/e$ of its amplitude after traversing a distance

$$1{\cdot}1\, V \{ 4\pi\mu h / \sigma p^3 \}^{\frac{1}{4}}.$$

When the alternations are faster than a certain limit depending upon the specific resistance of the material, the magnetic permeability, and the distance between the plates, the vibrations are always propagated with the velocity of light. Equation (38*) shows that they cannot, under any circumstances, be propagated faster than this.

Let us now find the quantity of electricity on the inside of the plate. If Σ be the surface density, we have

$$V^2 4\pi \Sigma = \left[m \left\{ A\epsilon^{mh} - B\epsilon^{-mh} \right\} + mC\epsilon^{-mh} \right] \epsilon^{imz}\, \epsilon^{ipt}$$
$$= 2mA\epsilon^{mh}\, \epsilon^{imz}\, \epsilon^{ipt}.$$

But, from equation (33),

$$E\kappa \{ 1 + hn \} = \frac{m^2}{p} A \text{ approximately} \ldots\ldots\ldots\ldots\ldots (41),$$

$$2\kappa h E = G\epsilon^{-nh} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42).$$

Let Γ = total current crossing unit breadth of a section of the conductor on the positive side of the origin, made by a plane at right

angles to the axis of z; then

$$\dot{\Gamma} = -\frac{1}{\sigma}\int_h^\infty \left\{\frac{dH}{dt} + \frac{d\phi}{dz}\right\} dx$$

$$= -\frac{ip}{\sigma n} G\epsilon^{-nh} \epsilon^{imz} \epsilon^{ipt}.$$

Hence, substituting, we get

$$4\pi \Sigma V^2 = \frac{i\Gamma n\sigma}{hm}\{1+hn\}.$$

But $m = p/V$ approximately, so that

$$4\pi\Sigma = \frac{i\Gamma n\sigma}{Vhp}(1+hn).$$

This gives the charge in terms of the current. If we express it in terms of the electromotive intensity at the point, we see that it shows that the charge per unit area at each point is the same as if the opposite plates had been kept respectively at potentials differing by $\{1+hn\}\, n\, V/p$ times the electromotive force at the edge of the plate. Thus, if n be so large that hn is large, compared with unity, the charge on the plates is independent of the distance between them.

We can investigate the resistance of the conductor in the way given by Lord Rayleigh in his paper on "The Self-induction and Resistance of Straight Conductors," *Phil. Mag.*, May, 1886.

Current across a strip of the conductor of width l,

$$= -\frac{ip}{\sigma n} G\left\{\epsilon^{-nh} - \epsilon^{-n(h+l)}\right\},$$

$$= -\frac{ip}{\sigma n} G\epsilon^{-nh}\, nl \text{ if } nl \text{ be small,}$$

$$= -\frac{ip}{\sigma n} G\epsilon^{-nh} \text{ if } nl \text{ be large,}$$

$$l\left\{\text{mean value of } \frac{d\phi}{dz} \text{ over this strip}\right\}$$

$$= iC\left\{\epsilon^{-mh} - \epsilon^{-m(h+l)}\right\}.$$

Now m is, in general, very small compared with n, and the only case that is of any practical importance is the one where ml is small, so

the mean value $= iC\epsilon^{-mh}ml,$

so that
$$\frac{\text{mean value of } d\phi/dz}{\text{current across the strip}} = -\frac{\sigma}{pl}\frac{C\epsilon^{-mh}m}{G\epsilon^{-nh}}.$$

Now we can easily prove, from equations (41), (42), and (27), that

$$\frac{C\epsilon^{-mh}}{G\epsilon^{-nh}} = \frac{p\{1+hn\}}{m},$$

so that, when nl is small,

$$\frac{\text{mean value of } d\phi/dz}{\text{current across the strip}} = -\frac{\sigma}{l}\{1+hn\} = -\frac{\sigma}{l} \text{ if } nh \text{ be small.}$$

So that the resistance of the strip is σ/l, as we should have expected.

If nl be large, we have

$$\frac{\text{mean value of } d\phi/dz}{\text{current across the strip}} = -\sigma nm\frac{G\epsilon^{-nh}}{C\epsilon^{-mh}}$$
$$= -\sigma n\{1+hn\}$$
$$= -\{2\pi\mu p\sigma\}^{\frac{1}{2}} - i\left[\{2\pi\mu p\sigma\}^{\frac{1}{2}} + 4\pi\mu ph\right],$$

so that the resistance $= \{2\pi\mu p\sigma\}^{\frac{1}{2}}$,

the self-induction $= \left\{\frac{2\pi\mu\sigma}{p}\right\}^{\frac{1}{2}} + 4\pi\mu h.$

These results agree with those obtained by Lord Rayleigh in the paper mentioned before.

The above investigation shows that all very rapid oscillations are propagated with the velocity of light along a conducting wire, and that in most cases oscillations whose period is as large as the $\frac{1}{100}$ of a second are propagated with this velocity; the exception to this is the case of a copper cable with considerable capacity, where the velocity of propagation is smaller. It also shows that the distance a disturbance must travel before its amplitude falls to $1/e$ of its original value, is very much greater for an iron cable than for a copper one of similar dimensions.

Formula for the interchange of the Independent and Dependent Variables in a Differential Expression; with extensions of the same, and some applications to Reciprocants. By C. LEUDESDORF, M.A.

[*Read June* 10*th*, 1886.]

1. Let x and y be two variables connected in any way; let y_1, y_2, &c. denote $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. and x_1, x_2, &c. denote $\frac{dx}{dy}$, $\frac{d^2x}{dy^2}$, &c. In what follows, $\frac{dy}{dx}$ often occupies a somewhat exceptional position compared with the other differential coefficients of y with respect to x, and it is then convenient to have a special symbol for it distinct in character from these; when this is the case it will be denoted by t. Similarly, when convenient, $\frac{dx}{dy}$ will be denoted by τ. The *weights* of x_r and y_r will in all cases be taken to be r. By V will always be denoted the operator

$$3y_2^2\frac{d}{dy_3}+10y_2y_3\frac{d}{dy_4}+(15y_2y_4+10y_3^2)\frac{d}{dy_5}+\text{\&c.},$$

which plays such an important part in the Theory of Reciprocants. If τ, x_2, x_3, &c. be expressed in terms of t, y_2, y_3, &c., we have

$$\begin{aligned}
\tau &= 1 && \div t,\\
x_2 &= -y_2 && \div t^3,\\
x_3 &= -ty_3+3y_2^2 && \div t^5,\\
x_4 &= -t^2y_4+10ty_2y_3-15y_2^3 && \div t^7,
\end{aligned}$$

and so on. Let the numerators on the right-hand side be denoted by Y_1, Y_2, Y_3, &c.; and let them be made homogeneous in t and q by suitably inserting powers of $q=1$, so that

$$Y_n = t^{n-2}A_0+t^{n-3}qA_1+t^{n-4}q^2A_2+\text{\&c.},$$

where A_0, A_1, &c. are homogeneous isobaric functions of y_2, y_3, &c., and do not involve t. Then it has been shown in a previous paper of mine (*Proceedings*, Vol. XVII., p. 197) that

$$VY_n = -tq\frac{dY_n}{dq} \quad\text{..............................}(1),$$

or, what is the same thing,

$$VY_n = t^2\frac{dY_n}{dt} - (n-2)\,tY_n \quad \text{.....................(2)};$$

and, as an immediate consequence, that

$$VA_0 = -A_1,\ \ VA_1 = -2A_2\ \ldots,\ \ VA_{n-3} = -(n-2)\,A_{n-2},\ \ VA_{n-2} = 0 \quad \text{............(3)}.$$

2. The formulæ at the end of the last paragraph may be arrived at independently of the results of the previous paper alluded to. They are easily seen to hold for the values 2, 3, 4 of n; suppose, now, that (1) or (2) holds for any particular value of n, it may then be proved to hold for the next value of n. For this purpose it is necessary to make use of the formula, due, I believe, to Captain MacMahon,

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)f = (w+i)\,y_2 f \quad \text{..................(4)},$$

where f is a rational integral homogeneous function of y_2, y_3, &c., of degree i and weight w. As y_r is taken to have the weight r, Y_n will be of degree $n-1$ and weight $2(n-1)$; consequently A_r will be of degree $r+1$ and weight $n+r$. Thus (4) gives

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)A_r = (n+2r+1)\,y_2A_r,$$

and therefore

$$\begin{aligned}\left(V\frac{d}{dx} - \frac{d}{dx}V\right)Y_n &= y_2\left\{(n+1)\,A_0t^{n-2} + (n+3)\,A_1t^{n-3}q + \ldots\right\} \\ &= y_2\left\{(n+1)\,Y_n + 2\frac{dY_n}{dq}\right\} \quad \text{.....................(5)}.\end{aligned}$$

But it is easily seen that

$$Y_{n+1} = t\frac{dY_n}{dx} - (2n-1)\,y_2Y_n \quad \text{.....................(6)},$$

from which, combined with (5),

$$\begin{aligned}V\left\{Y_{n+1} + (2n-1)\,y_2Y_n\right\} - t\frac{d}{dx}(VY_n) &= (n+1)\,y_2tY_n + 2y_2t\frac{dY_n}{dq} \\ &= (n+1)\,y_2tY_n - 2y_2VY_n,\end{aligned}$$

by (1); therefore

$$VY_{n+1} = t\frac{d}{dx}(VY_n) + (n+1)\,ty_2Y_n - (2n+1)\,y_2VY_n \quad \text{......(7)}.$$

Now, by (2), $\quad VY_n = t^2\dfrac{dY_n}{dt} - (n-2)\,tY_n,$

therefore

$$\frac{d}{dx}(VY_n) = 2ty_2\frac{dY_n}{dt} + t^2\frac{d}{dx}\left(\frac{dY_n}{dt}\right) - (n-2)\,y_2Y_n - (n-2)\,t\,\frac{dY_n}{dx}$$

$$= 2ty_2\frac{dY_n}{dt} + t^2\frac{d}{dt}\left(\frac{dY_n}{dx}\right) - (n-2)\,y_2Y_n - (n-2)\,t\,\frac{dY_n}{dx},$$

which, on substituting for $\frac{dY_n}{dx}$ from (6), becomes

$$= t\frac{dY_{n+1}}{dt} + (2n+1)\,y_2t\frac{dY_n}{dt} - \{Y_{n+1} + (2n-1)\,y_2Y_n\}$$

$$-(n-2)\,y_2Y_n - (n-2)\,\{Y_{n+1} + (2n-1)\,y_2Y_n\}$$

$$= t\frac{dY_{n+1}}{dt} + (2n+1)\,y_2t\frac{dY_n}{dt} - (n-1)\,Y_{n+1} - (2n^2-2n-1)\,y_2Y_n,$$

and if this be substituted for $\frac{d}{dx}(VY_n)$, in (7), we find

$$VY_{n+1} = t^2\frac{dY_{n+1}}{dt} - (n-1)\,tY_{n+1};$$

showing that the result (2) is true when $n+1$ is written for n; and so, by induction, is true for all positive integral values of n.

3. From (3), we have

$$Y_n = t^{n-2}A_0 - t^{n-3}qVA_0 + \frac{1}{1.2}t^{n-4}q^2V^2A_0 - \dots$$

$$= t^{n-2}e^{-V/t}A_0$$

$$= -t^{n-2}e^{-V/t}y_n,\text{*}$$

---

* Otherwise: Since

$$VY_n = -tq\frac{d}{dq}Y_n,$$

$$V^2Y_n = -V\left(tq\frac{dY_n}{dq}\right)$$

$$= -tq\frac{d}{dq}(VY_n)$$

$$= t^2q^2\frac{d^2Y_n}{dq^2},$$

since V and $\frac{d}{dq}$ are evidently commutative, and V acts neither on t nor on q.

In a similar manner, if r be any integer,

$$V^rY_n = (-tq)^r\frac{d^rY_n}{dq^r},$$

and therefore, since $x_n = Y_n t^{-(2n-1)}$,

$$x_n = -t^{-(n+1)} e^{-\frac{V}{t}} y_n \qquad\qquad (8),$$

a formula which gives the value of any differential coefficient of x with respect to y in terms of differential coefficients of y with respect to x.

4. The result (8) may at once be extended. For, since

$$Y_m = -t^{m-2} y_m + t^{m-3} q V y_m - \frac{t^{m-4}}{1.2} q^2 V^2 y_m + \&c.,$$

$$Y_n = -t^{n-2} y_n + t^{n-3} q V y_n - \frac{t^{n-4}}{1.2} q^2 V^2 y_n + \&c.,$$

therefore

$$\begin{aligned} Y_m Y_n = t^{m+n-4} y_m y_n - t^{m+n-5} q\,(y_m V y_n + y_n V y_m) & \\ + \frac{t^{m+n-6}}{1.2} q^2\,(y_m V^2 y_n + 2 V y_m V y_n + y_n V^2 y_m) & \\ - \frac{t^{m+n-7}}{1.2.3} q^3 (y_m V^3 y_n + 3 V y_m V^2 y_n + 3 V^2 y_m V y_n + y_n V^3 y_m) & \\ + \&c. & \end{aligned}$$

$$= t^{m+n-4} e^{-\frac{V}{t}} (y_m y_n);$$

therefore

$$\begin{aligned} Y_m^{\alpha} Y_n^{\beta} Y_p^{\gamma} \ldots &= (-1)^{\alpha+\beta+\gamma+\ldots}\, t^{(m-2)\alpha}\, t^{(n-2)\beta}\, t^{(p-2)\gamma} \ldots e^{-\frac{V}{t}} (y_m^{\alpha} y_n^{\beta} y_p^{\gamma} \ldots) \\ &= (-1)^i t^{w-2i} e^{-\frac{V}{t}} (y_m^{\alpha} y_n^{\beta} y_p^{\gamma} \ldots), \end{aligned}$$

if i be the degree and w the weight of $y_m^{\alpha} y_n^{\beta} y_p^{\gamma} \ldots$.

---

so that, if $f(\theta)$ be any function which can be expanded in powers of θ,

$$f\left(\frac{V}{t}\right) Y_n = f\left(-q \frac{d}{dq}\right) Y_n.$$

In particular, $e^{\frac{V}{t}} Y_n = e^{-q\frac{d}{dq}} Y_n$

$=$ result of writing $q-q$ or zero instead of q in Y_n

$= -t^{n-2} y_n$;

therefore $\qquad Y_n = -t^{n-2} e^{-\frac{V}{t}} y_n,$

as before.

Accordingly, if f denote any homogeneous isobaric function of degree i and weight w,

$$f(Y_2, Y_3, \ldots) = (-1)^i t^{w-2i} e^{-\frac{V}{t}} f(y_2, y_3, \ldots).$$

But $$f(Y_2, Y_3, \ldots Y_n) = f(x_2 t^3, x_3 t^5, \ldots x_n t^{2n-1})$$
$$= t^{2w-i} f(x_2, x_3, \ldots x_n),$$

therefore $$f(x_2, x_3, \ldots) = (-1)^i t^{-(w+i)} e^{-\frac{V}{t}} f(y_2, y_3, \ldots) \quad \text{.........(9)},$$

a formula which gives the means of expressing what a homogeneous isobaric function of y_2, y_3, &c. transforms into when x and y are written in place of y and x respectively.

5. Now let

$$f(t, y_2, y_3, \ldots) = C_0 t^n + C_1 t^{n-1} + C_2 t^{n-2} + \&c. \quad \text{............(10)}$$

be any homogeneous and isobaric function of degree i and weight w; C_0, C_1, &c. being functions of y_2, y_3, &c. (and not involving t), of degrees $i-n$, $i-(n-1)$, &c., and of weights $w-n$, $w-(n-1)$, &c. Let us examine the effect of writing x for y and y for x in f. By (9),

$$C_0 \text{ will become } (-1)^{i-n} t^{-(w+i-2n)} e^{-\frac{V}{t}} C_0,$$

$$C_1 \quad \text{,,} \quad \text{,,} \quad (-1)^{i-(n-1)} t^{-(w+i-2n+2)} e^{-\frac{V}{t}} C_1,$$

and so on; also t will be changed into τ. Accordingly, the right-hand expression in (10) will become

$$(-1)^{i-n} t^{-(w+i)} e^{-\frac{V}{t}} \{C_0 t^{2n} \tau^n - C_1 t^{2(n-1)} \tau^{n-1} + \&c.\},$$

or $$(-1)^{i-n} t^{-(w+i)} e^{-\frac{V}{t}} \{C_0 t^n - C_1 t^{n-1} + \&c.\},$$

so that we have

$$f(\tau, x_2, x_3, \ldots) = (-1)^i t^{-(w+i)} e^{-\frac{V}{t}} f(-t, y_2, y_3, \ldots) \quad \text{......(11)},$$

the complete formula, which includes (8) and (9), and which gives the means of expressing what a homogeneous isobaric function of y_1, y_2, y_3, &c. becomes when x and y are written one in place of the other. This formula will, therefore, serve to effect the interchange of the independent and dependent variables in any algebraical function whatever of the differential coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c.

6. Evidently the process of § 3, by which (8) was deduced from (1), could be worked backwards so as to deduce (1) from (8); *i.e.*, each of these equations implies the other, and (3) is a direct consequence of (1). In a precisely similar way, then, we may deduce from (9) that, if any homogeneous isobaric function of x_2, x_3, &c. be expressed in terms of t, y_2, y_3, &c., and be written in the form

$$f = C_0 t^n + C_1 t^{n-1} q + C_2 t^{n-2} q^2 + \ldots + C_n,$$

where C_0, C_1, &c. are functions of y_2, y_3, &c., and do not involve t, and $q = 1$ is inserted for the sake of homogeneity; then

$$Vf = -t\frac{df}{dq},$$

and $VC_0 = -C_1,\quad VC_1 = -2C_2,\quad VC_2 = -3C_3,\ \ldots,\quad VC_n = 0.$

7. The effect upon a differential expression of writing, in place of the variables, general linear functions of these, may now be found. Writing $\xi = ax+by+c$, $\eta = a'x+b'y+c'$, where a, b, c, a', b', c' are constants, we have, if η_1, η_2, η_3, ... denote the successive differential coefficients of η with regard to ξ,

$$\eta_1 = \frac{a'+b't}{a+bt},$$

$$\eta_2 = \frac{a'b-ab'}{b^3}\cdot\frac{-y_2}{\left(t+\dfrac{a}{b}\right)^3},$$

$$\eta_3 = \frac{a'b-ab'}{b^4}\cdot\frac{-\left(t+\dfrac{a}{b}\right)y_3+3y_2^2}{\left(t+\dfrac{a}{b}\right)^5},$$

and so on.

Comparing these results with those given in § 1, it is seen that η_n may (except in the case $n = 1$) be derived from x_n by simply writing $t+\dfrac{a}{b}$ in place of t, and multiplying by the factor $\dfrac{a'b-ab'}{b^{n+1}}$.

Proceeding, then, exactly as in § 3, we shall obtain the formula

$$\eta_n = (ab'-a'b)(tb+a)^{-(n+1)} e^{-\frac{V}{t+a/b}} y_n \quad \ldots\ldots\ldots\ldots\ldots(12),$$

which again, just as in § 4, will lead to the formula

$$f(\eta_2, \eta_3, \ldots \eta_n) = (ab'-a'b)^i (tb+a)^{-(w+i)} e^{-\frac{V}{t+a/b}} f(y_2, y_3, \ldots y_n) \quad \ldots(13),$$

analogous to (9); and then to the formula

$$f(\eta_1, \eta_2, \eta_3, \ldots) = (ab'-a'b)^i (tb+a)^{-(w+i)} e^{-\frac{V}{t+a/b}}$$
$$\times f\left\{\frac{(tb+a)(tb'+a')}{ab'-a'b}, y_2, y_3, \ldots\right\} \ldots\ldots\ldots(14),$$

analogous to (11).

The result (14) enables us to find the effect made by a general linear change of the variables upon any homogeneous algebraic function of the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, &c. As the change of x into y and y into x is only a very particular case of the general linear substitution, it is, of course, seen that the formulæ of §§ 3, 4, 5 are all included in the general result (14).

8. Two other particular cases of (14) are of interest.

(1) The orthogonal substitution

$$\xi = x\cos\theta - y\sin\theta, \quad \eta = x\sin\theta + y\cos\theta.$$

Here we have $a = b' = \cos\theta$, $a' = -b = \sin\theta$, and (14) becomes

$$f(\eta_1, \eta_2, \eta_3, \ldots)$$
$$= (\cos\theta - t\sin\theta)^{-(w+i)} e^{-\frac{V}{t-\cot\theta}} f\left\{\frac{1-t^2}{2}\sin 2\theta + t\cos 2\theta, y_2, y_3, \ldots\right\} \ldots(15).$$

(2) The imaginary substitution $\xi + \eta\sqrt{-1} = x$, $\eta + \xi\sqrt{-1} = y$, or $\xi = \frac{1}{2}(x - y\sqrt{-1})$, $\eta = \frac{1}{2}(y - x\sqrt{-1})$.

Here we have $a = b' = \frac{1}{2}$, $a' = b = -\frac{1}{2}\sqrt{-1}$, so that (14) becomes

$$f(\eta_1, \eta_2, \eta_3, \ldots) = \frac{2^w}{(1-t\sqrt{-1})^{w+i}} e^{-\frac{V}{t+\sqrt{-1}}} f\left\{\frac{1+t^2}{2\sqrt{-1}}, y_2, y_3, \ldots\right\} \ldots(16),$$

the symbol f denoting in all these cases a homogeneous isobaric function.

I now proceed to some applications of the foregoing results to Reciprocants.

9. In equation (9), suppose $Vf = 0$; then

$$f(x_2, x_3, \ldots) = (-1)^i t^{-(w+i)} f(y_2, y_3, \ldots);$$

that is to say, f must be a pure reciprocant. Conversely, if f is a homogeneous isobaric pure reciprocant,

$$e^{-\frac{V}{t}} f = f,$$

so that Vf must be zero.

10. Consider any expression (of degree i and weight w)

$$\phi = A_0 t^n + A_1 t^{n-1} + A_2 t^{n-2} + \dots + A_n \quad \dots\dots\dots\dots\dots (17),$$

where $A_0, A_1, \dots A_n$ are homogeneous isobaric functions of $y_2, y_3, \dots$ and do not involve t; and are, moreover, subject to the conditions

$$VA_0 = -A_1, \quad VA_1 = -2A_2, \quad VA_2 = -3A_3, \ \dots, \ VA_n = 0.$$

Then, by (9), if we write x for y and y for x in $A_0 t^n$, it will become

$$(-1)^i t^{-(w+i)} e^{-\frac{V}{t}} (A_0 t^n),$$

that is, $$(-1)^i t^{-(w+i)} \phi.$$

But, with the same transformation, ϕ will become, by (11),

$$(-1)^i t^{-(w+i)} (-1)^n e^{-\frac{V}{t}} \{A_0 t^n - A_1 t^{n-1} + \dots\},$$

that is, $$(-1)^i t^{-(w+i)} (-1)^n A_0 t^n.$$

It follows that, if we multiply $A_0 t^n$ by any power of t, and add it to (or subtract it from) ϕ, we shall have a reciprocant. For the expression so formed will not change when x and y are interchanged, except for a power of t and a possible change of sign. In particular, the expressions

$$\pm A_0 t^n + A_0 t^n + A_1 t^{n-1} + \dots + A_n \quad \dots\dots\dots\dots\dots\dots (18)$$

are both of them homogeneous reciprocants.

A particular case of a function of the form (17) was considered in my previous paper on Reciprocants, already quoted. The results there obtained with regard to this case may be extended to all functions of the form (17), which may be called "quasi-covariants," for reasons explained in that paper.

11. Any mixed homogeneous reciprocant of degree i and weight w can be written in the form (10), so that

$$R(t, y_2, y_3, \dots) = C_0 t^n + C_1 t^{n-1} + \dots + C_n.$$

Then, since R is a reciprocant,

$$R(\tau, x_2, x_3, \dots) = (-1)^{i-n} t^{-(w+i)} R(t, y_2, y_3, \dots).$$

Therefore, by (11),

$$R(t, y_2, y_3, \dots) = (-1)^{-n} e^{-\frac{V}{t}} R(-t, y_2, y_3, \dots) \quad \dots\dots\dots (19).$$

Therefore $C_0t^n + C_1t^{n-1} + C_2t^{n-2} + \dots + C_n$

$$= e^{-\frac{V}{t}}\left\{C_0t^n - C_1t^{n-1} + C_2t^{n-2} - \dots + (-1)^n C_n\right\}$$

$$\begin{aligned}= C_0t^n - C_1t^{n-1} + C_2t^{n-2} - C_3t^{n-3} + \dots \\ - VC_0t^{n-1} + VC_1t^{n-2} - VC_2t^{n-3} + \dots \\ + \frac{1}{1.2}V^2C_0t^{n-2} - \frac{1}{1.2}V^2C_1t^{n-3} + \dots \\ - \frac{1}{1.2.3}V^3C_0t^{n-3} + \dots \\ + \&c.\end{aligned}$$

Equating coefficients of the various powers of t, we find

$$\begin{aligned} C_0 &= C_0, \\ -C_1 &= C_1 + VC_0, \\ C_2 &= C_2 + VC_1 + \frac{1}{1.2}V^2C_0, \\ -C_3 &= C_3 + VC_2 + \frac{1}{1.2}V^2C_1 + \frac{1}{1.2.3}V^3C_0, \\ &\&c., \quad \&c.,\end{aligned}$$

which reduce to

$$\left.\begin{aligned} -2C_1 &= VC_0, \\ -2C_3 &= VC_2 + \frac{1}{1.2}V^2C_1 + \frac{1}{1.2.3}V^3C_0, \\ -2C_5 &= VC_4 + \frac{1}{1.2}V^2C_3 + \frac{1}{1.2.3}V^3C_2 \\ &\quad + \frac{1}{1.2.3.4}V^4C_1 + \frac{1}{1.2.3.4.5}V^5C_0, \\ -2C_7 &= \&c. \end{aligned}\right\} \quad \dots (20),$$

and so on.

These relations (20) are therefore the necessary and sufficient conditions which must be satisfied by the coefficients of the powers of t in any mixed homogeneous reciprocant whatever, when written in the form R above.

12. The relations (20) are not, in general, sufficient to determine C_1, C_2, &c. in terms of C_0; so that it is not possible to write down any general form which a mixed homogeneous reciprocant must have.

A very large number of such reciprocants are of one or other of the forms (18), which may, I think, with propriety be called the *standard* forms.

It can now be shown that every mixed homogeneous reciprocant is either itself of one of the standard forms, or else may be expressed as a sum of such standard forms of the same character.

(a) Take first the case of a reciprocant R of negative character. We may write

$$\begin{aligned} R &= C_0t^n+C_1t^{n-1}+C_2t^{n-2}+\dots \\ &= 2t^nA_0-t^{n-1}VA_0+\frac{t^{n-2}}{1.2}V^2A_0-\frac{t^{n-3}}{1.2.3}V^3A_0+\frac{t^{n-4}}{1.2.3.4}V^4A_0-\&\text{c.} \\ &\qquad\qquad + 2t^{n-2}A_0' - t^{n-3}VA_0' + \frac{t^{n-4}}{1.2}V^2A_0'-\&\text{c.} \\ &\qquad\qquad\qquad\qquad + 2t^{n-4}A_0''-\&\text{c.}, \end{aligned}$$

and so on (where each horizontal row is a reciprocant of the standard form and of negative character), if only

$$\left.\begin{aligned} C_0 &= 2A_0, \\ C_1 &= -VA_0, \\ C_2 &= 2A_0'+\frac{1}{1.2}V^2A_0, \\ C_3 &= -VA_0'-\frac{1}{1.2.3}V^3A_0', \\ C_4 &= 2A_0''+\frac{1}{1.2}V^2A_0'+\frac{1}{1.2.3.4}V^4A_0, \\ &\&\text{c.},\quad \&\text{c.}, \end{aligned}\right\}\dots\dots\dots\dots(21),$$

or

$$\left.\begin{aligned} 2A_0 &= C_0, \\ 2A_0' &= C_2+\frac{1}{1.2}VC_1, \\ 2A_0'' &= C_4+\frac{3}{1.2}VC_3+\frac{1}{1.2}V^2C_2+\frac{1}{1.2.3.4}V^3C_1, \\ &\&\text{c.},\quad \&\text{c.} \end{aligned}\right\}\dots\dots\dots(22).$$

But if the values of C_0, C_1, &c., as given in (21), be substituted in the equations of condition (20), they will be found to satisfy them *identically*. It follows that R can always be expressed as a sum of standard negative homogeneous reciprocants; we have, in fact, only

to choose for A_0, A_0', A_0'', &c. such functions of C_0, C_1, C_2, &c. as are given by the equations (22).

(*b*) Similarly, the case of a reciprocant R of positive character may be dealt with. We may write

$$\begin{aligned} R &= C_0 t^n + C_1 t^{n-1} + C_2 t^{n-2} + \dots \\ &= t^n A_1 - \frac{t^{n-1}}{\lfloor 2} V A_1 + \frac{t^{n-2}}{\lfloor 3} V^2 A_1 - \frac{t^{n-3}}{\lfloor 4} V^3 A_1 + \frac{t^{n-4}}{\lfloor 5} V^4 A_1 - \&\text{c.} \\ &\qquad + t^{n-2} A_1' - \frac{t^{n-3}}{\lfloor 2} V A_1' + \frac{t^{n-4}}{\lfloor 3} V^2 A_1' - \&\text{c.} \\ &\qquad\qquad + t^{n-4} A_1'' - \&\text{c.}, \end{aligned}$$

and so on (each horizontal row being a reciprocant of standard form and positive character), provided that

$$\left.\begin{aligned} C_0 &= A_1, \\ C_1 &= -\frac{1}{2} V A_1, \\ C_2 &= A_1' + \frac{1}{1.2.3} V^2 A_1, \\ C_3 &= -\frac{1}{1.2} V A_1' - \frac{1}{1.2.3.4} V^3 A_1', \\ C_4 &= A_1'' + \frac{1}{1.2.3} V^2 A_1' + \frac{1}{1.2.3.4.5} V^4 A_1, \\ C_5 &= -\frac{1}{1.2} V A_1'' - \frac{1}{1.2.3.4} V^3 A_1' - \frac{1}{1.2.3.4.5.6} V^5 A_1, \end{aligned}\right\} \dots\dots (23),$$

and so on, or

$$\left.\begin{aligned} A_1 &= C_0, \\ A_1' &= C_2 + \frac{2}{1.2.3} V C_1, \\ A_1'' &= C_4 + \frac{4}{1.2.3} V C_3 + \frac{1}{1.2.3} V^2 C_2 + \frac{2}{1.2.3.4.5} V^3 C_1, \\ &\&\text{c.}, \quad \&\text{c.} \end{aligned}\right\} \dots\dots (24).$$

If now the values of C_0, C_1, &c., as given in (23), be substituted in equations (20), they will be found to satisfy them identically. It follows that, in this case also, R can be expressed as a sum of standard (positive) homogeneous reciprocants; it is only necessary, in fact, to

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choose for A_1, A'_1, A''_1, &c., their values in terms of C_0, C_1, &c., as given by equations (24).

13. Suppose, if possible, a homogeneous reciprocant R of such a kind that it involves only even (or only odd) powers of t. Then, if t^n be the highest power of t which occurs in it,

$$R(-t, y_2, y_3, \ldots) = (-1)^n R(t, y_2, y_3, \ldots),$$

and (19) becomes

$$R(t, y_2, y_3, \ldots) = e^{-\frac{V}{t}} R(t, y_2, y_3, \ldots),$$

or
$$C_0 t^n + C_1 t^{n-1} + C_2 t^{n-2} + \ldots = C_0 t^n + C_1 t^{n-1} + C_2 t^{n-2} + \&c.$$
$$- VC_0 t^{n-1} - VC_1 t^{n-2} - \&c.$$
$$+ \frac{1}{1.2} V^2 C_0 t^{n-2} + \&c.$$
$$- \&c.$$

Therefore
$$C_0 = C_0,$$
$$C_1 = C_1 - VC_0,$$
$$C_2 = C_2 - VC_1 + \frac{1}{1.2} V^2 C_0,$$

and so on; whence $VC_0 = 0$, $VC_1 = 0$, &c., *i.e.*, C_0, C_1, &c. must all be pure reciprocants.

The conclusion is, that there can be no mixed homogeneous reciprocant which involves t only in even (or only in odd) powers, except such an one as is made up of the sum of a number of pure reciprocants each multiplied by some even (odd) power of t.

14. The following results will be useful later (in § 15).

With the same notation as in § 10, we have

$$\phi(t) = t^n e^{-\frac{V}{t}} A_0,$$

therefore
$$e^{-\frac{V}{t}} \phi(t) = e^{-\frac{2V}{t}} (t^n A_0),$$
$$= 2^n \phi\left(\frac{t}{2}\right),$$

therefore $e^{-\frac{V}{t}} \left[\pm A_0 t^n + A_0 t^n + A_1 t^{n-1} + \ldots + A_n\right] = 2^n \phi\left(\frac{t}{2}\right) \pm \phi(t).$

In a similar manner, if

$$\phi(pq) = A_0 (pq)^n + A_1 (pq)^{n-1} + \&c.,$$

where p, q are any functions of t,

$$e^{-\frac{V}{p}}\phi(pq) = e^{-\frac{V}{p}}(pq)^n e^{-\frac{V}{pq}} A_0$$

$$= e^{-\frac{1+q}{pq}V} A_0 (pq)^n$$

$$= (1+q)^n \phi\left(\frac{pq}{1+q}\right),$$

and
$$e^{-\frac{V}{p}} A_0 (pq)^n = q^n e^{-\frac{V}{p}} A_0 p^n$$

$$= q^n \phi(p),$$

therefore $e^{-\frac{V}{p}}\left[\pm A_0 (pq)^n + A_0 (pq)^n + A_1 (pq)^{n-1} + \ldots + A_n\right]$

$$= (1+q)^n \phi\left(\frac{pq}{1+q}\right) \pm q^n \phi(p) \quad \text{......(25).}$$

15. By help of (25), the effect of the general linear substitution

$$\xi = ax+by+c, \quad \eta = a'x+b'y+c'$$

on a mixed homogeneous reciprocant of standard form (18) may easily be expressed. For, if in (14) we write

$$t+\frac{a}{b} = p, \quad \frac{t+\dfrac{a'}{b'}}{\dfrac{a}{b}-\dfrac{a'}{b'}} = q,$$

and therefore
$$t+\frac{a'}{b'} = \frac{pq}{1+q},$$

and then make use of (25) to simplify the expression on the right of (14), it is seen that the result of the substitution is to change the reciprocant $\pm A_0 t^n + \phi(t)$, where

$$\phi(t) = A_0 t^n + A_1 t^{n-1} + \&c.,$$

into

$$\frac{(bb')^n (ab'-a'b)^{i-n}}{(tb+a)^{w+i}}\left[\left(t+\frac{a}{b}\right)^n \phi\left(t+\frac{a'}{b'}\right) \pm \left(t+\frac{a'}{b'}\right)^n \phi\left(t+\frac{a}{b}\right)\right]$$

$$\text{............(26).}$$

16. For the particular case of the imaginary transformation

$$\xi = \tfrac{1}{2}(x-y\sqrt{-1}), \quad \eta = \tfrac{1}{2}(y-x\sqrt{-1}),$$

the expression (26) becomes

$$\frac{2^w}{(2\sqrt{-1})^n}\left(\frac{\sqrt{-1}}{t+\sqrt{-1}}\right)^{w+i}$$
$$\times\left[(t+\sqrt{-1})^n\phi(t-\sqrt{-1})\pm(t-\sqrt{-1})^n\phi(t+\sqrt{-1})\right],$$

or, as it may be written,

$$2^{w-n}\frac{(t-\sqrt{-1})^n}{(1-t\sqrt{-1})^{w+i-n}}\left[\left(A_0+\frac{A_1}{t+\sqrt{-1}}+\frac{A_2}{(t+\sqrt{-1})^2}+\&c.\right)\right.$$
$$\left.\pm\left(A_0+\frac{A_1}{t-\sqrt{-1}}+\frac{A_2}{(t-\sqrt{-1})^2}+\&c.\right)\right]\ldots(27).$$

We may make use of this to verify a theorem of Mr. Rogers', viz., that if the imaginary substitution above be made in a mixed homogeneous reciprocant, it will transform it into an orthogonal reciprocant. Bearing in mind the result of § 12, that any such mixed homogeneous reciprocant can be written as the sum of a number of standard ones of the same character, and also the fact that a reciprocant made up of the sum of any number of orthogonal reciprocants of the same character must be an *orthogonal* reciprocant, it is seen that all that is required is to show that the expression within the square brackets in (27) is an *orthogonal* reciprocant. A reciprocant it must evidently be.

17. To prove this, I recall that, as shown in my previous paper, in order that any reciprocant F should be an orthogonal reciprocant, it is necessary and sufficient that UF should be equal to some numerical multiple of tF, where

$$UF=(1+t^2)\frac{dF}{dt}+\left(3y_2\frac{dF}{dy_2}+4y_3\frac{dF}{dy_3}+\ldots\right)t+VF.$$

Now, since A_0 is of degree $i-n$ and weight $w-n$, A_1 of degree $i-(n-1)$ and weight $w-(n-1)$, &c., therefore

$$UA_0=VA_0+(w+i-2n)\,tA_0\qquad=-A_1+(w+i-2n)\,tA_0,$$

$$UA_1=VA_1+(w+i-2n+2)\,tA_1=-2A_2+(w+i-2n+2)\,tA_1,$$

and so on; also $\qquad U(t+\sqrt{-1})=1+t^2.$

Therefore

$$U\frac{A_1}{t+\sqrt{-1}} = \frac{(t+\sqrt{-1})\,UA_1 - A_1(1+t^2)}{(t+\sqrt{-1})^2}$$

$$= \frac{UA_1 - A_1(t-\sqrt{-1})}{t+\sqrt{-1}}$$

$$= \frac{-2A_2 + (w+i-2n)\,tA_1}{t+\sqrt{-1}} + A_1$$

$$U\frac{A_2}{(t+\sqrt{-1})^2} = \frac{UA_2 - 2A_2(t-\sqrt{-1})}{(t+\sqrt{-1})^2}$$

$$= \frac{-3A_3 + (w+i-2n)\,tA_2}{(t+\sqrt{-1})^2} + \frac{2A_2}{t+\sqrt{-1}},$$

and so on. Therefore, by addition,

$$U\left[A_0 + \frac{A_1}{t+\sqrt{-1}} + \frac{A_2}{(t+\sqrt{-1})^2} + \dots\right]$$

$$= (w+i-2n)\,t\left[A_0 + \frac{A_1}{t+\sqrt{-1}} + \frac{A_2}{(t+\sqrt{-1})^2} + \dots\right] \dots (28).$$

Similarly, it could be shown that

$$U\left[A_0 + \frac{A_1}{t-\sqrt{-1}} + \frac{A_2}{(t-\sqrt{-1})^2} + \dots\right]$$

$$= (w+i-2n)\,t\left[A_0 + \frac{A_1}{t+\sqrt{-1}} + \frac{A_2}{(t-\sqrt{-1})^2} + \dots\right] \dots (29).$$

If, then, F denote the expression in the square brackets in (27), it follows, from (28) and (29), by addition or subtraction, that

$$UF = (w+i-2n)\,tF;$$

i.e., F is an orthogonal reciprocant, and is moreover such that it is made absolute by the factor $y_2^{-\frac{w+i-2n}{3}}$, or the factor $(1+t^2)^{-\left(\frac{w+i}{2}-n\right)}$. Mr. Rogers' theorem has, therefore, been verified.

Second Paper on Reciprocants. *By* L. J. Rogers, B.A.

[*Read June* 10*th*, 1886.]

§ 1. In my last memoir on Reciprocants, pp. 220–231, I showed the existence of a certain class of mixed homogeneous reciprocants, to which I gave the name of homographic.

I propose in the following pages to show how all such reciprocants can be very conveniently expressed in terms of a certain series of protomorphs slightly differing from the series M_1, M_2, M_3 ... which I used in my last memoir.

Homographic reciprocants are, as has already been shown, annihilated by the operator

$$t\delta_a + 3a\delta_b + 6b\delta_c + 10c\delta_d + \ldots \quad \text{.......................(1)},$$

which I call H.

Let G_m denote the mixed generator

$$t\delta_x - \frac{n}{2}a \quad \text{...............................(2)},$$

where n is the characteristic of the reciprocant to be operated upon. Then we know that $G_m M_1 = M_2$, $G_m M_2 = M_3$, &c., and generally $G_m M_m = M_{m+1}$. Also the characteristic of M_m is $3m$.

Now, let N_1, N_2, N_3 ... be a set of protomorphs of characteristic 3, 6, 9 ... respectively, of which the successive formation is given by the general equation

$$N_{m+1} = \lambda_m G_m N_m + \mu_m a^2 N_{m-1} \quad \text{....................(3)},$$

where λ_m, μ_m are numerical and functions of m. It is always possible to suppose this, as the characteristic of N_{m+1} is $3m+3$, and therefore $6+$that of N_{m-1}; and that of a^2 being six also, we see that N_{m+1} and $a^2 N_{m-1}$ are of the same characteristic. The character of the three terms in the equation is likewise seen to be the same.

Let also, if possible, the operation of H on N_{m+1} give $k_m N_m t$ as result, k_m being numerical. By hypothesis, then,

$$HN_{m+1} = k_m N_m t \text{.............................(4)}.$$

Operating on (3) with H, we therefore get

$$k_m N_m t = \lambda_m HG_m N_m + 2\mu_m a t N_{m-1} + \mu_m t a^2 k_{m-2} N_{m-2} \text{.........(5)}.$$

We now have to evaluate $HG_m N_m$.

It is easy to show that

$$H\delta_x - \delta_x H = t\delta_t + 2a\delta_a + 3b\delta_b + \dots.$$

We know, too, that

$$(2t\delta_t + 3a\delta_a + 4b\delta_b + \dots)\, N_m = 3mN_m,$$

and that $$(t\delta_t + a\delta_a + \dots)\, N_m = mN_m.$$

Hence $$(H\delta_x - \delta_x H)\, N_m = 2mN_m,$$

therefore

$$\begin{aligned} H\delta_x N_m &= \delta_x H N_m + 2mN_m \\ &= k_{m-1}\delta_x (N_{m-1}t) + 2mN_m, \text{ by (4)}, \\ &= k_{m-1} a N_{m-1} + k_{m-1} t\delta_x N_{m-1} + 2mN_m \\ &= k_{m-1} a N_{m-1} + k_{m-1} G_m N_{m-1} + k_{m-1}\frac{3m-3}{2} a N_{m-1} + 2mN_m, \text{ by (2)}, \\ &= k_{m-1}\left(G_m N_{m-1} + \frac{3m-1}{2} a N_{m-1}\right) + 2mN_m \quad \dots\dots\dots\dots(6). \end{aligned}$$

Again, $$\begin{aligned} HG_m N_m &= tH\delta_x N_m - \frac{3m}{2} tN_m - \frac{3m}{2} atk_{m-1} N_{m-1} \\ &= tk_{m-1}\left\{G_m N_{m-1} - \frac{1}{2} a N_{m-1}\right\} + \frac{m}{2} tN_m \quad \dots\dots(7), \end{aligned}$$

as we finally get after substituting for $H\delta_x N_m$ in (6). But

$$G_m N_{m-1} = \frac{N_m - \mu_{m-1} a^2 N_{m-2}}{\lambda_{m-1}}, \text{ by (3)},$$

therefore, substituting in (7), we get

$$\begin{aligned} k_m t N_m = \lambda\ \ tk_{m-1}\left(\frac{N_m - \mu_{m-1} a^2 N_{m-2}}{\lambda_{m-1}} - \frac{1}{2} a N_{m-1}\right) + \lambda_m \frac{m}{2} tN_m \\ + 2\mu_m atN_{m-1} + \mu_m a^2 tk_{m-2} N_{m-2} \dots\dots\dots\dots(8), \end{aligned}$$

an equation which ought identically to be true. Hence the coefficient of N_m is zero, since N_m contains linearly a letter which N_{m-1}, N_{m-2} do not contain. Therefore

$$\frac{k_m}{\lambda_m} - \frac{k_{m-1}}{\lambda_{m-1}} = \frac{m}{2} \quad \dots\dots\dots\dots(9).$$

Similarly, equating coefficients of N_{m-1}, N_{m-2}, we get

$$\frac{\lambda_m k_{m-1}}{\mu_m} = 4 \quad\text{.......................................}(10),$$

and
$$\frac{\lambda_m k_{m-1}}{\mu_m} = \frac{\lambda_{m-1} k_{m-2}}{\mu_{m-1}},$$

which are not independent, since the second follows from the first.

Now, we are free to give λ_m any value we please. Let this value be 2. Then (9) becomes

$$k_m - k_{m-1} = m,$$

and (10)
$$\mu_m = \tfrac{1}{2} k_{m-1}.$$

It is most convenient to take N_1, N_2 as the same as M_1, M_2, *i.e.*, a and $tb - \frac{3}{2}a^2$, whence
$$HN_1 = t,$$
$$HN_2 = 0;$$

therefore $k_1 = 0$, $k_2 = 2$, $k_3 = 5$, $k_4 = 9$, &c. ...;

and, generally,
$$k_m = \tfrac{1}{2}(m-1)(m+2),$$
$$\mu_m = \tfrac{1}{4}(m-2)(m+1).$$

Hence, if
$$N_1 = a,$$
$$N_2 = tb - \tfrac{3}{2}a^2,$$
$$N_3 = 2G_m N_2,$$
$$N_4 = 2G_m N_3 + a^2 N_2,$$
$$N_5 = 2G_m N_4 + \tfrac{5}{2}a^2 N_3,$$
$$N_6 = 2G_m N_5 + \tfrac{9}{2}a^2 N_4,$$
&c. ...,

then $HN_2 = 0$, $HN_3 = 2tN_2$, $HN_4 = 5tN_3$, ... &c.,

which is the same as saying

$$H = t\left(2N_2 \frac{d}{dN_3} + 5N_3 \frac{d}{dN_4} + 9N_4 \frac{d}{dN_5} + \dots\right) \quad\text{.............}(11),$$

when we operate on any function of these N's. Whence we see that, when homographic reciprocants are expressed in terms of the protomorphs N, they assume a binariant form.

This manner of forming such a series of protomorphs that homo-

graphic reciprocants could be written as a binariant function of them, was suggested to me by a very similar theorem relating to Projective Reciprocants, put forward by Professor Sylvester and Mr. Hammond, —a correction of a theorem of mine I had supposed true, but they had discovered to be erroneous.

Having obtained the above annihilator (11) for homographic reciprocants, we can deduce a similar one for the analogous orthogonals, viz., the circular reciprocants.

Let $$G_0 = (1+t^2)\,\delta_x - at,$$

the orthogonal generator. Then, if

$$\begin{aligned}
\psi_1 &= a,\\
\psi_2 &= (1+t^2)\,b - 3a^2t,\\
\psi_3 &= G_0\psi_2,\\
\psi_4 &= G_0\psi_3 - a^2\psi_2,\\
\psi_5 &= G_0\psi_4 - \tfrac{5}{2}a^2\psi_3,\\
&\text{\&c.,} \qquad \text{\&c.,}
\end{aligned}$$

the annihilator for circular reciprocants will be

$$2\psi_2\frac{d}{d\psi_3} + 5\psi_3\frac{d}{d\psi_4} + 9\psi_4\frac{d}{d\psi_5} + \dots \qquad \dots\dots\dots\dots\dots(12),$$

when they are expressed as a function of the ψ's as above defined.

In order to point out more clearly the similarity between the theorems established above and that of Professor Sylvester and Mr. Hammond, I have thought it advisable to introduce here a proof of the latter theorem, without adopting the numerical alterations they make, viz., writing $1.2.a$ for a, $1.2.3.b$ for b, &c., and consequently $a\delta_b + 2b\delta_c + 3c\delta_d + \dots$ for the projective annihilator Ω.

§ 2. In the unsimplified form

$$\Omega \equiv 3a\delta_b + 8b\delta_c + 15c\delta_d + \dots \qquad \dots\dots\dots\dots\dots\dots(1)$$

the differences between each successive coefficient forming the arithmetic series $\qquad 5,\ 7,\ 9,\ \dots.$

Let G_p denote the pure generator

$$a\delta_x - \frac{n}{3}\,b \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots(2),$$

and, as in my former memoir on Reciprocants,

$$R_2 = ac - \tfrac{5}{3}b^2,$$

$$R_3 = a^2d - 5abc + \tfrac{40}{9}b^3,$$

and, generally, $$R = G_p R_{m-1} \text{ .... (3).}$$

Here n, the characteristic, is four times the weight.

This series of simplest educts is not, however, convenient for expressing projective reciprocants.

As in § 1, let us take another series of protomorphs S_2, S_3, S_4, ... of weights $8.12.16$... such that

$$\Omega S_{m+1} = k_m a S_m \text{ .... (4),}$$

where k_m is numerical, and if possible such that

$$S_{m+1} = \lambda_m G_p S_m + \mu_m R_2 S_{m-1} \text{ .... (5).}$$

Operating on each side with Ω and substituting in (4), and remembering that

$$\Omega\delta_x S_m - \delta_x \Omega S_m = (3a\delta_a + 5b\delta_b + \ldots)\, S_m = (2n-3m)\, S_m = 5mS_m,$$

we get, proceeding step by step in the same manner as in § 1,

$$\frac{k_m}{\lambda_m} - \frac{k_{m-1}}{\lambda_{m-1}} = m \text{ .... (6),}$$

$$\frac{k_{m-2}\lambda_{m-1}}{\mu_{m-1}} = \frac{k_{m-1}\lambda_m}{\mu_m} = -6.$$

Assuming S_3 to be the same as R_3, we know that

$$\Omega S_3 = 0,$$

i.e., $$k_2 = 0.$$

Let $\lambda\ = 1$, so that $$k_m - k\ _{-1} = m,$$

$$k_3 = 3,$$

$$k_4 = 7,$$

$$k_5 = 12, \qquad \text{\&c. ...}$$

$$k_m = \tfrac{1}{2}(m-2)(m+3),$$

and $$\mu_m = -\tfrac{1}{6}k_{m-} = -\tfrac{1}{12}(m-3)(m+2).$$

Hence, if
$$S_3 = R_3,$$
$$S_4 = G_p S_3,$$
$$S_5 = G_p S_4 - \tfrac{1}{2} R_2 S_3 = R_5 - \tfrac{1}{2} R_2 R_3,$$
$$S_6 = G_p S_5 - \tfrac{7}{6} R_2 S_4,$$
&c.,

then
$$\Omega \equiv a\left(3S_3 \frac{d}{dS_4} + 7S_4 \frac{d}{dS_5} + 12 S_5 \frac{d}{dS_6} + \dots\right) \dots\dots\dots\dots (7).$$

And any function of the S's annihilated by Ω, as given in (7), will be a projective reciprocant.

The simplest projective after S_3 is
$$6S_3 S_5 - 7S_4^2,$$
which is easily seen to be the same as
$$6R_3 R_5 - 7R_4^2 - 3R_3^2 R_2 \dots\dots\dots\dots\dots\dots\dots\dots (8),$$
which is of weight 8, order 8, and characteristic 32.

We can obtain a series of projective protomorphs in two ways, either by forming them on the model of ordinary invariants and altering the numerical coefficients so as to make them admit of annihilation by Ω (7), and so get them expressed in terms of the S's; or by forming an absolute reciprocant and differentiating. If we differentiate with respect to x, we get a very complicated result, but if with respect to r, where $r = \int a^{\frac{1}{3}} dx$, we have, as shown in my last memoir, $\dfrac{dR_m}{dr} = R_{m+1}$, so that the production of successive protomorphs is very simple. We thus get them in terms of the R's.

For instance, from R_3 and $6R_3R_5 - 7R_4^2 - 3R_3^2 R_2$ can be deduced a projective reciprocant with $R_3^2 R_6$ for its first term, of characteristic 48. From the latter and from R_3, we deduce another beginning with $R_3^3 R_7$, of characteristic 64, and so on.

§ 3. The method of obtaining a third reciprocant from two given ones admits of easy explanation. For the sake of brevity, let us take the following statement:—
$$S \text{ is rec. } n\ (q),$$
as meaning, S is a reciprocant of characteristic n and character q; q being ± 1.

If, then, R is rec. m (p),

and S is rec. n (q),

then will $mR\delta_x S - nS\delta_x R$ be rec. $m+n+1$, (pq).

For $\dfrac{S^m}{R^n}$ is rec. 0, $(p^n q^m)$,

whence by taking logarithms and differentiating, we get that

$$\frac{mR\delta_x S - nS\delta_x R}{RS} \text{ is rec. } 1, (+).$$

Hence $mR\delta_x S - nS\delta_x R$ is rec. $m+n+1$, (pq) ....(1).

When we are treating of M, ϕ, or R functions, we can put this expression into a similar form. For instance, R and S being two R-functions of weights m and n, we get

$$4mR\delta_x S - 4nS\delta_x R$$

as a reciprocant, by (1), since the characteristic is four times the weight.

That is, $(mR\delta_r S - nS\delta_r R)\dfrac{dr}{dx}$ is rec.

But $\dfrac{dr}{dx} = a^{\frac{1}{3}}$ and is a reciprocant. Hence

$mR\delta_r S - nS\delta_r R$ is rec. ....(2),

of characteristic $4(m+n+1)$, since the weight is $m+n+1$.

On a Class of Projective Reciprocants.

§ 4. Let (ξ, η) be the Boothian coordinates of any tangent to a given curve, and let (x, y) be the Cartesian coordinates of the point of contact.

Then it is well known that three relations exist between ξ, η, x, and y, viz.,

$$\left.\begin{aligned} x\xi + y\eta &= 1 \\ xd\xi + yd\eta &= 0 \\ \xi dx + \eta dy &= 0 \end{aligned}\right\} \quad \text{..........................(1)},$$

Let r be a function of x and y, and ρ the same function of ξ, η. The most convenient function to take for the value r will be determined hereafter.

Let x_1, x_2, &c., y_1, y_2, &c., denote $\frac{dx}{dr}$, $\frac{d^2x}{dr^2}$, &c., with similar abbreviations for the Greek letters. Then

$$\left.\begin{aligned} x\xi_1+y\eta_1 &= 0 \\ \xi x_1+\eta y_1 &= 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$x\xi_2+y\eta_2+(x_1\xi_1+y_1\eta_1)\frac{dr}{d\rho}=0,$$

$$\xi x_2+\eta y_2+(x_1\xi_1+y_1\eta_1)\frac{d\rho}{dr}=0.$$

Eliminating $\frac{dr}{d\rho}$, we get

$$(x\xi_2+y\eta_2)(\xi x_2+\eta y_2)=(x_1\xi_1+y_1\eta_1)^2,$$

or $$\frac{\eta y}{x_1\xi_1}(\eta_2\xi_1-\xi_2\eta_1)(y_2x_1-x_2y_1)=(x_1\xi_1+y_1\eta_1)^2, \text{ by (2).}$$

From (2) and (1), we also have

$$x_1\xi_1+y_1\eta_1=x_1\xi_1\left(1+\frac{x\xi}{y\eta}\right)=\frac{x_1\xi_1}{y\eta}.$$

Hence $$(\eta_2\xi_1-\xi_2\eta_1)(y_2x_1-x_2y_1)=(x_1\xi_1+y_1\eta_1)^3 \quad\ldots\ldots\ldots\ldots(3).$$

Now it is easy to show, by elementary Differential Calculus, that

$$y_2x_1-x_2y_1=\frac{d^2y}{dx^2}\left(\frac{dx}{dr}\right)^3.$$

It is, therefore, best to define r as such a function that

$$\frac{d^2y}{dx^2}=\left(\frac{dr}{dx}\right)^3 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

so that r is the same function as has been used in the preceding sections. And for ρ that $\frac{d^2\eta}{d\xi^2}=\left(\frac{d\rho}{d\xi}\right)^3.$

Whence (3) becomes while

$$\left.\begin{aligned} x_1\xi_1+y_1\eta_1 &= 1 \\ y_2x_1-x_2y_1 &= 1 \\ \eta_2\xi_1-\xi_2\eta_1 &= 1 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

In terms of r and x we may very conveniently express the series of pure reciprocant educts R_2, R_3, R_4, ..., though now it is more con-

venient to consider them as the *absolute* pure reciprocants, instead of only the numerators of such. For, by (4),

$$a^{-\frac{1}{3}} = x_1,$$

$$x_2 = -\tfrac{1}{3} b a^{-\frac{5}{3}},$$

$$x_3 = -\tfrac{1}{3}\left(c a^{-\frac{5}{3}} - \tfrac{5}{3} b^2 a^{-\frac{8}{3}}\right) a^{-\frac{1}{3}} = -\tfrac{1}{3} R_2 x_1.$$

Hence $$R_2 = -3x_3/x_1,$$

and this, by differentiating this second equation in (5), $= -3y_3/y_1$.

Again, $$R_3 = \frac{1}{a^{\frac{1}{3}}} \frac{d}{dx} R_2 = \frac{d}{dr} R_2,$$

and generally $$R_n = \frac{d}{dr} R_{n-1} \quad \text{.............................(6)}.$$

We shall denote by P_2, P_3, ... the corresponding absolute reciprocants in (ξ, η).

Also $r_1, r_2, \ldots$ will stand for $\dfrac{dr}{d\rho} \cdot \dfrac{d^2 r}{d\rho^2} \ldots,$

and $\rho_1, \rho_2, \ldots$,, $\dfrac{d\rho}{dr} \cdot \dfrac{d^2\rho}{dr^2} \ldots.$

By expressing R_2, R_3, ... in terms of r and ρ, we shall find the connection between the R's and the P's.

By (5), we see that, for the value we fix for R, the equations following (2) become $$x\xi_2 + y\eta_2 = -\frac{dr}{d\rho},$$

which, by (2) and (5), gives

$$\frac{dr}{d\rho} = \frac{x}{\eta_1} = -\frac{y}{\xi_1}, \text{ by (2).}$$

By symmetry $$\frac{d\rho}{dr} = \frac{\xi}{y_1} = -\frac{\eta}{x_1},$$

therefore $$\frac{d\rho}{dr}(y_1 x - x_1 y) = \xi x + \eta y = 1 \quad \text{by (1)},$$

therefore $$r_1 = y_1 x - x_1 y \quad \text{...............................(7)}.$$

Differentiating for ρ, we have

$$\frac{r_2}{r_1} = y_2x - x_2y,$$

$$\frac{r_3r_1 - r_2^2}{r_1^3} = y_3x - x_3y + y_2x_1 - x_2y_1$$

$$= y_3x - x_3y + 1 \quad \text{by (5)}$$

$$= x_3\left(\frac{y_1}{x_1}x - y\right) + 1 \quad \text{because } \frac{y_3}{y_1} = \frac{x_3}{x_1} = -\tfrac{1}{3}R_2$$

$$= \frac{x_3}{x_1}r_1 + 1 = -\tfrac{1}{3}R_2r_1 + 1,$$

therefore $$-\tfrac{1}{3}R_2 = \frac{r_3r_1 - r_2^2}{r_1^4} - \frac{1}{r_1} \quad \text{........................(8).}$$

Differentiating again for ρ, by (6), we have

$$-\tfrac{1}{3}R_3 = \frac{r_4r_1 - 5r_3r_2}{r_1^5} + \frac{4r_2^3}{r_1^6} - \frac{r_2}{r_1^3} \quad \text{.....................(9).}$$

Now, the right side of (9) is a reciprocant in r and ρ, since the numerator of the first term is the post-Schwarzian in r and ρ, and the other terms are well-known forms.

Since, then, by symmetry,

$$-\tfrac{1}{3}\mathrm{P}_3 = \frac{\rho_4\rho_1 - 5\rho_3\rho_2}{\rho_1^5} + \frac{4\rho_2^3}{\rho_1^6} - \frac{\rho_2}{\rho_1^3},$$

we have, finally, $$R_3 = -\mathrm{P}_3\rho_1^3 \quad \text{..............................(10).}$$

Thus R_3 is an odd reciprocant in r and ρ, with characteristic 3. Let us write this statement thus, as in § 3,

R_3 is rec. 3, $(-)$.

Now ρ_2 „ 3, $(-)$.

Hence $R_4\rho_2 - R_3\rho_3$ is rec. 7, $(+)$, by § 3......(11),

$3R_3(R_5\rho_2 - R_3\rho_4) - 7R_4(R_4\rho_2 - R_3\rho_3)$ „ 11, $(-)$ ....(12).

Again, $5R_4^2\rho_2^2 - 10R_4R_3\rho_3\rho_2 + 5R_3^2\rho_3^2$ is rec. 14, $(+)$, by (11),

and $R_3^2(3\rho_2\rho_4 - 5\rho_3^2)$ „ 14, $(+)$.

Adding, we get, after dividing by ρ_2 (3 $-$),

$$5R_4^2\rho_2-10R_4R_3\rho_3+3R_3^2\rho_4 \text{ is rec. } 11, (-) \text{ ....(13).}$$

Eliminating the term $R_4R_3\rho_3$ from (12) and (13), we get

$$(30R_3R_5-35R_4^2)\,\rho_2-9R_3^2\rho_4 \text{ is rec. } 11, (-)\text{...........(14).}$$

Referring to (8), we easily get

$$-\tfrac{1}{3}R_2 = -\frac{\rho_3\rho_1-2\rho_2^2}{\rho_1^2}-\rho_1,$$

therefore $$\rho_4-\tfrac{5}{3}R_2\rho_2 = \frac{\rho_1\rho_4-5\rho_3\rho_2}{\rho_1}+\tfrac{10}{3}\frac{\rho_2^3}{\rho_1^2}-\tfrac{5}{3}\rho_1\rho_2,$$

which is recip. 5, $(-)$.

Hence $$R_3^2(\rho_4-\tfrac{5}{3}R_2\rho_2) \text{ is rec. } 11, (-).$$

Eliminating $R_3^2\rho_4$ by means of (14), we get, after dividing by $5\rho_2$,

$$6R_3R_5-7R_4^2-3R_3^2R_2 \text{ is rec. } 8, (+) \text{ ....(15),}$$

i.e., $$= (6\mathrm{P}_3\mathrm{P}_5-7\mathrm{P}_4^2-3\mathrm{P}_3^2\mathrm{P}_2)\,\rho_1^8.$$

We have now obtained two R-functions, *i.e.*, two absolute pure reciprocants in x and y, which are also reciprocants for r and ρ. We have therefore one absolute reciprocant for r and ρ, and can by differentiation obtain an infinite number of them.

This series of reciprocants is of high importance geometrically. They lead when integrated, as Mr. Hammond has pointed out, to equations of curves whose polar reciprocals with respect to any point are curves of the same kind.

Thus, $$R_3 = 0$$

leads to the general equation to a conic. Transforming into Boothians, we have, by (10), $$\mathrm{P}_3 = 0,$$

i.e., the polar reciprocal is also a conic.

Let us therefore call these reciprocants self-polar. All such reciprocants are projective. For we know that $6R_3R_5-7R_4^2-3R_3^2R_2$ and R_3 are projective, and that their weights are in the ratio of 3 to 8. So also, treated as reciprocants in r and ρ, are their characteristics in the ratio of 3 to 8. Employing the methods of § 3 for the formation of new reciprocants, it is evident that we shall form a system of self-polar protomorphs identical with the projective protomorphs obtained at the end of § 2.

We see, then, that all self-polar reciprocants are projective, although the converse is not true.

Some Applications of Weierstrass's Elliptic Functions.

By Mr. A. G. GREENHILL.

[*Read June* 10*th*, 1886.]

In this paper it is proposed to exhibit the use of Weierstrass's Elliptic Functions, by showing their direct application to several geometrical and physical problems, and thus to give illustrations of the meaning of the analytical formulæ expressing the relations between these functions.

The formulæ are, in general, quoted without demonstration, and applied immediately to the problem requiring their use; the reader, however, who is desirous of following out the rigorous demonstration of these formulæ by the methods of pure mathematics, is recommended to consult Schwarz's *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen;* or Halphen's *Traité des Fonctions Elliptiques et de leurs Applications*, Paris, 1886.

I. *Confocal Cartesians, and Quartic Curves.*

1. Let
$$z = \wp u,$$
where
$$z = x+iy, \quad u = \tfrac{1}{2}(\xi+i\eta),$$
and $\wp u$ denotes Weierstrass's elliptic function of u, defined by the equation
$$u = \int_z^\infty \frac{dz}{\sqrt{(4z^3-g_2z-g_3)}};$$
so that
$$z = \wp u,$$
and
$$\left(\frac{dz}{du}\right)^2 = (\wp' u)^2$$
$$= 4z^3-g_2z-g_3$$
$$= 4(z-e_1)(z-e_2)(z-e_3), \text{ suppose.}$$

Then, if e_1, e_2, e_3 are all real, they will define the positions of the three foci F_1, F_2, F_3 on the axis of x of a system of confocal Cartesians, given by the equations
$$\xi = \text{const.}, \quad \text{and} \quad \eta = \text{const.},$$
from the relation
$$x+iy = \wp\, \tfrac{1}{2}(\xi+i\eta);$$
and, from the properties of *conjugate functions*, it follows immediately that these confocal Cartesians intersect at right angles.

2. With the notation of the *sigma functions* explained by Schwarz or Halphen, we put

$$\wp u - e_1 = \left(\frac{\sigma_1 u}{\sigma u}\right)^2,$$

$$\wp u - e_2 = \left(\frac{\sigma_2 u}{\sigma u}\right)^2,$$

$$\wp u - e_3 = \left(\frac{\sigma_3 u}{\sigma u}\right)^2.$$

In the ordinary notation of elliptic functions,

$$\wp u - e_1 = (e_1 - e_3)\frac{\mathrm{cn}^2 \sqrt{(e_1-e_3)}\, u}{\mathrm{sn}^2 \sqrt{(e_1-e_3)}\, u}, \quad \text{or } (e_1-e_3)\,\mathrm{cs}^2 \sqrt{(e_1-e_3)}\, u,$$

$$\wp u - e_2 = (e_1 - e_3)\frac{\mathrm{dn}^2 \sqrt{(e_1-e_3)}\, u}{\mathrm{sn}^2 \sqrt{(e_1-e_3)}\, u}, \quad \text{or } (e_1-e_3)\,\mathrm{ds}^2 \sqrt{(e_1-e_3)}\, u,$$

$$\wp u - e_3 = (e_1 - e_3)\frac{1}{\mathrm{sn}^2 \sqrt{(e_1-e_3)}\, u}, \quad \text{or } (e_1-e_3)\,\mathrm{ns}^2 \sqrt{(e_1-e_3)}\, u,$$

with
$$k^2 = \frac{e_2-e_3}{e_1-e_3}, \quad k'^2 = \frac{e_1-e_2}{e_1-e_3},$$

supposing $e_1 > e_2 > e_3$.

3. Denoting by r_1, r_2, r_3 the distances of a point P whose coordinates are x, y from the three foci F_1, F_2, F_3, and denoting $\frac{1}{2}(\xi - i\eta)$ by v; then

$$r_1 = \frac{\sigma_1 u}{\sigma\ u}\frac{\sigma_1 v}{\sigma\ v}, \quad r_2 = \frac{\sigma_2 u}{\sigma\ u}\frac{\sigma_2 v}{\sigma\ v}, \quad r_3 = \frac{\sigma_3 u}{\sigma\ u}\frac{\sigma_3 v}{\sigma\ v};$$

and, expressed in a real form by means of formula [*D*][9], p. 51, of Schwarz's *Formeln*,

$$r_1 = \frac{\sigma_1 u}{\sigma\ u}\frac{\sigma_1 v}{\sigma\ v}$$

$$= \frac{\sigma_1 u\ \sigma_2 u\ \sigma_1 v\ \sigma_2 v}{\sigma\ u\ \sigma_2 u\ \sigma\ v\ \sigma_2 v}$$

$$= -(e_3 - e_1)\frac{\sigma_1\xi\ \sigma_2 i\eta + \sigma_2\xi\ \sigma_1 i\eta}{\sigma_3\xi\ \sigma_1 i\eta - \sigma_1\xi\ \sigma_3 i\eta} \quad \text{..................(i.)};$$

or, again,
$$r_1 = \frac{\sigma_1 u\ \sigma_3 u\ \sigma_1 v\ \sigma_3 v}{\sigma\ u\ \sigma_3 u\ \sigma\ v\ \sigma_3 v}$$

$$= -(e_1-e_2)\,\frac{\sigma_1\xi\;\sigma_3 i\eta+\sigma_3\xi\;\sigma_1 i\eta}{\sigma_1\xi\;\sigma_2 i\eta-\sigma_2\xi\;\sigma_1 i\eta}\;\ldots\ldots\ldots\ldots\ldots\text{(ii.)}.$$

Similarly
$$r_2 = -(e_1-e_2)\,\frac{\sigma_2\xi\;\sigma_3 i\eta+\sigma_3\xi\;\sigma_2 i\eta}{\sigma_1\xi\;\sigma_2 i\eta-\sigma_2\xi\;\sigma_1 i\eta}\ldots\ldots\ldots\ldots\ldots\text{(iii.)},$$

$$= -(e_2-e_3)\,\frac{\sigma_1\xi\;\sigma_2 i\eta+\sigma_2\xi\;\sigma_1 i\eta}{\sigma_2\xi\;\sigma_3 i\eta-\sigma_3\xi\;\sigma_2 i\eta}\ldots\ldots\ldots\ldots\ldots\text{(iv.)},$$

and
$$r_3 = -(e_2-e_3)\,\frac{\sigma_3\xi\;\sigma_1 i\eta+\sigma_1\xi\;\sigma_3 i\eta}{\sigma_2\xi\;\sigma_3 i\eta-\sigma_3\xi\;\sigma_2 i\eta}\;\ldots\ldots\ldots\ldots\ldots\text{(v.)},$$

$$= -(e_3-e_1)\,\frac{\sigma_2\xi\;\sigma_3 i\eta+\sigma_3\xi\;\sigma_2 i\eta}{\sigma_3\xi\;\sigma_1 i\eta-\sigma_1\xi\;\sigma_3 i\eta}\ldots\ldots\ldots\ldots\ldots\text{(vi.)}.$$

4. Therefore, from (iv.) and (v.),

$$\left.\begin{aligned} r_2\sigma_3\,\xi-r_3\sigma_2\,\xi &= (e_2-e_3)\,\sigma_1\xi \\ r_2\sigma_3 i\eta-r_3\sigma_2 i\eta &= -(e_2-e_3)\,\sigma_1 i\eta \end{aligned}\right\}\;\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

the vectorial equations of conjugate confocal Cartesians; and from the remaining equations, by cyclical interchange of suffixes, we obtain

$$\left.\begin{aligned} r_3\,\sigma_1\xi-r_1\,\sigma_3\xi &= (e_3-e_1)\,\sigma_2\xi \\ r_3\sigma_1 i\eta-r_1\sigma_3 i\eta &= -(e_3-e_1)\,\sigma_2 i\eta \end{aligned}\right\}\;\ldots\ldots\ldots\ldots\ldots\text{(B)},$$

and
$$\left.\begin{aligned} r_1\,\sigma_2\xi-r_2\,\sigma_1\xi &= (e_1-e_2)\,\sigma_3\xi \\ r_1\sigma_2 i\eta-r_2\sigma_1 i\eta &= -(e_1-e_2)\,\sigma_3 i\eta \end{aligned}\right\}\;\ldots\ldots\ldots\ldots\ldots\text{(C)},$$

also
$$\left.\begin{aligned} (e_2-e_3)\,r_1\sigma_1\,\xi+(e_3-e_1)\,r_2\sigma_2\,\xi+(e_1-e_2)\,r_3\sigma_3\,\xi &= 0 \\ (e_2-e_3)\,r_1\sigma_1 i\eta+(e_3-e_1)\,r_2\sigma_2 i\eta+(e_1-e_2)\,r_3\sigma_3 i\eta &= 0 \end{aligned}\right\}\;\ldots\text{(D)};$$

and (A), (B), (C), (D) are the vectorial equation of the same confocal Cartesians in a symmetrical form (Darboux, *Annales Scientifiques de l'École Normale Supérieure*, Tome IV., 1867).

5. To indicate the values of the *invariants* g_2 and g_3, the notation

$$z = \wp\,(u\,;\;g_2,\,g_3)$$

is sometimes employed; and then it follows, from considerations of homogeneity, that

$$\begin{aligned} \wp\,(mu\,;\;g_2,\,g_3) &= \frac{1}{m^2}\,\wp\,(u\,;\;m^4g_2,\,m^6g_3), \\ \sigma\;(mu\,;\;g_2,\,g_3) &= m\sigma\;(u\,;\;m^4g_2,\,m^6g_3), \\ \sigma_\lambda\,(mu\,;\;g_2,\,g_3) &= \sigma_\lambda\,(u\,;\;m^4g_2,\,m^6g_3)\,; \end{aligned}$$

so that
$$\wp\,(i\eta\,;\ g_2, g_3) = -\wp\,(\eta\,;\ g_2, -g_3),$$
$$\sigma\,(i\eta\,;\ g_2, g_3) = \ i\sigma\,(\eta\,;\ g_2, -g_3),$$
$$\sigma_\lambda(i\eta\,;\ g_2, g_3) = \ \sigma_\lambda(\eta\,;\ g_2, -g_3)\,;$$

equivalent in Legendre and Jacobi's notation to a transformation to the complementary modulus.

If
$$\wp\omega_1 = e_1,\quad \wp\omega_2 = e_2,\quad \wp\omega_3 = e_3,$$

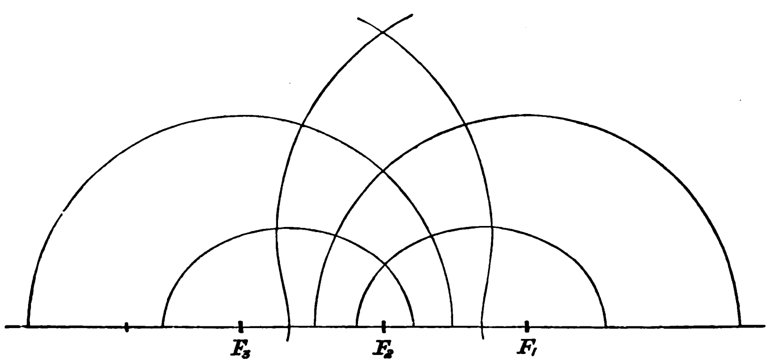
then ω_1, ω_2, ω_3 are called *half-periods* of the elliptic function $\wp u$; but of these only two are independent, as they are connected by the

relation
$$\omega_1+\omega_2+\omega_3 = 0;$$

also
$$e_1 + e_2 + e_3 = 0.$$

Supposing $e_1 > e_2 > e_3$, then ω_1 is real, but ω_3 is imaginary; and
$$\frac{\omega_3}{\omega_1} = i\frac{K'}{K},$$

where K and K' denote Jacobi's periods.



6. Then, when $\xi = \frac{1}{2}\omega_1$, the corresponding Cartesian is a circle, centre F_1, and containing F_2, F_3 being the corresponding point to F_2; and when $i\eta = \frac{1}{2}\omega_3$, the corresponding Cartesian is a circle, centre F_3, and containing F_2, F_1 being the corresponding point to F_3.

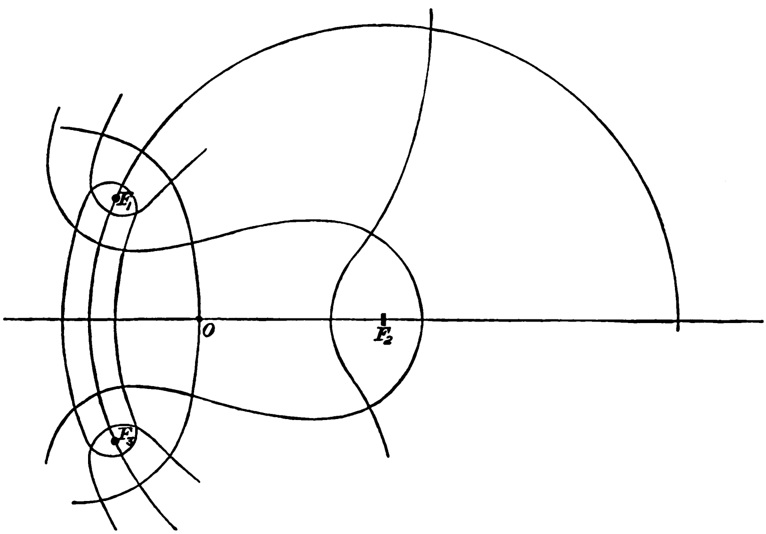
The two ovals of the same Cartesian are given by ξ and $\omega_1 - \xi$, or $i\eta$ and $\omega_3 - i\eta$.

7. If the *discriminant* $g_2^3 - 27g_3^2$ is negative, two of the quantities e_1, e_2, e_3 are imaginary; denoting them by e_1 and e_3, where $e_1 - e_3$ is positive imaginary, the corresponding foci form an isosceles triangle $F_1F_2F_3$, and the origin O is at the centre of gravity of the triangle, since

$$e_1 + e_2 + e_3 = 0.$$

Then $$x + iy = \wp \tfrac{1}{2} (\xi + i\eta)$$

denotes a series of orthogonal quartic curves, associated with Cartesians.



The values $\xi = \frac{1}{2}\omega_1$ or $i\eta = \frac{1}{2}\omega_3$ will each give $r_2 = F_1F_2$, so that the corresponding quartics double down into circular arcs of centre F_2, limited by F_1 and F_3.

(Holzmüller, *Einführung in die Theorie der isogonalen Verwandtschaften*).

If $g_3 = 0$, the triangle $F_1F_2F_3$ is equilateral.

8. Incidentally we notice the electrical application of these formulæ; namely, the electrification of an insulated cylinder whose cross section in one of these quartic curves is proportional to $(r_1r_2r_3)^{-\frac{1}{2}}$; and in particular, for a cross section the limited arc of a circle, the electrification is proportional to $(r_1r_3)^{-\frac{1}{2}}$, r_1 and r_3 denoting the distances of a point on the surface from the edges of the cylinder.

Then $\xi =$ const. and $\eta =$ const. will represent either the equi potential surfaces, or the orthogonal lines of force.

9. Consider the system given by

$$u = \int \frac{dz}{(1-z^3)^{\frac{2}{3}}}$$

(Siebeck, *Crelle*, 57 and 59, *Ueber eine Gattung von Curven vierten Grades, &c.*; Schwarz, *Crelle*, 77, *Ueber ebene algebraische Isothermen*).

Then
$$\frac{1+z}{1-z} = \wp'\left(\frac{u}{\sqrt[3]{3}};\ 0,\ \tfrac{1}{3}\right),$$

$$\frac{(1-z^3)^{\frac{2}{3}}}{1-z} = \sqrt{3}\,\wp\left(\frac{u}{\sqrt[3]{3}};\ 0,\ \tfrac{1}{3}\right);$$

or
$$\frac{1+z^3}{1-z^3} = \wp'(u;\ 0, -1),$$

$$\frac{z}{(1-z^3)^{\frac{2}{3}}} = \wp(u;\ 0, -1),$$

giving a system of sextic orthogonal curves.

II. *Reciprocants.*

10. Consider the Mixed Reciprocant

$$tc - 5ab = 0,$$

or
$$\frac{dy}{dx}\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2}\frac{d^3y}{dx^3} = 0,$$

given in Prof. Sylvester's Inaugural Lecture, Dec. 12, 1885, and published in *Nature*, Jan. 7, 1886.

Then
$$\frac{d^4y}{dx^4}\Big/\frac{d^3y}{dx^3} = 5\,\frac{d^2y}{dx^2}\Big/\frac{dy}{dx};$$

and, integrating,
$$\log\frac{d^3y}{dx^3} = 5\log\frac{dy}{dx} + \text{const.},$$

or
$$\frac{d^3y}{dx^3} = C\left(\frac{dy}{dx}\right)^5.$$

Multiplying by $\frac{d^2y}{dx^2}$, and integrating again,

$$\tfrac{1}{2}\left(\frac{d^2y}{dx^2}\right)^2 = \tfrac{1}{6}C\left(\frac{dy}{dx}\right)^6 + C;$$

or, changing the constants,

$$\left(\frac{dt}{dx}\right)^2 = \kappa t^6 + \lambda,$$

so that
$$x = \int \frac{dt}{\sqrt{(\kappa t^6+\lambda)}} + \mu,$$
$$y = \int \frac{t\,dt}{\sqrt{(\kappa t^6+\lambda)}} + \nu.$$

11. By a change of origin, we can make μ and ν vanish, and by orthogonal projection parallel to the axes we can reduce κ and λ to unity, so that we need only consider

$$x = \int \frac{dt}{\sqrt{(1+t^6)}}, \quad y = \int \frac{t\,dt}{\sqrt{(1+t^6)}} \quad \text{.................(i.)},$$

or
$$x = \int \frac{dt}{\sqrt{(1-t^6)}}, \quad y = \int \frac{t\,dt}{\sqrt{(1-t^6)}} \quad \text{..............(ii.)},$$

or
$$x = \int \frac{dt}{\sqrt{(t^6-1)}}, \quad y = \int \frac{t\,dt}{\sqrt{(t^6-1)}} \quad \text{..............(iii.)}.$$

12. From (i.), $x = -\int \frac{dt^{-2}}{\sqrt{(4t^{-6}+4)}}, \quad y = \int \frac{dt^2}{(4t^6+4)},$

so that
$$t^{-2} = \wp(x;\ 0, -4),$$
$$t^2 = \wp(y;\ 0, -4);$$

and therefore
$$\wp x\, \wp y = 1,$$

with $g_2 = 0$, $g_3 = -4$; representing, in figure (i.), a series of nearly circular curves round centres whose coordinates are $2m\omega_2$, $2m'\omega_2$; and a series of conjugate points at $(2m+1)\,\omega_2$, $(2m'+1)\,\omega_2$.

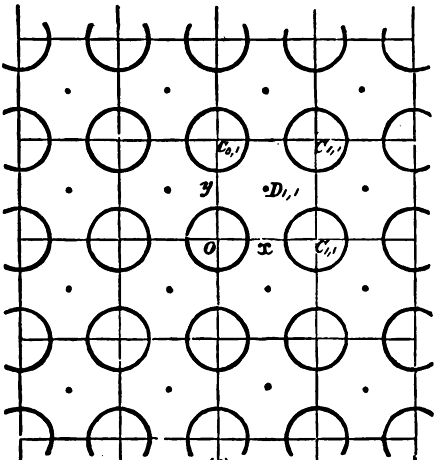
13. From (ii.),
$$x = -\int \frac{dt^{-2}}{\sqrt{(4t^{-6}-4)}}, \quad y = \int \frac{dt^2}{\sqrt{(-4t^6+4)}};$$

so that
$$t^{-2} = \wp(x;\ 0,\ 4),$$
$$t^2 = -\wp(y;\ 0, -4);$$

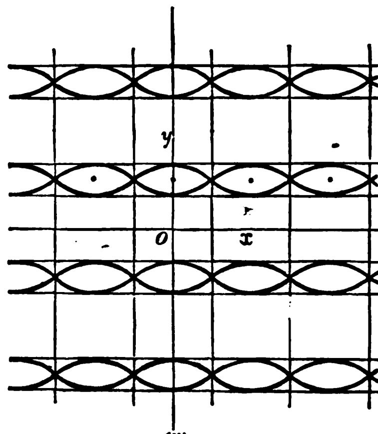
and therefore
$$\wp x\, \wp y = -1,$$

representing figure (ii.), and the figure of (iii.) is that of (ii.) turned through a right angle; having

$$t^{-2} = -\wp(x;\ 0, -4),$$
$$t^2 = \wp(y;\ 0,\ 4).$$



(i)



(ii)

14. The intrinsic equation of (i.) is

$$s = \int \frac{\sqrt{(1+t^2)}\,dt}{\sqrt{(1+t^6)}}$$

$$= \int \frac{dt}{\sqrt{(1-t^2+t^4)}}$$

$$= \int \frac{d\psi}{\sqrt{(1-\frac{3}{4}\sin^2 2\psi)}}, \text{ if } t = \tan\psi,$$

the elliptic integral of the first kind, to modulus sin 60°; so that

$$2\psi = \operatorname{am} 2s,$$

and
$$\sin 2\psi = \operatorname{sn} 2s,$$

the intrinsic equation of the curve.

It is curious that the modular angle in the Cartesian equation of the curve is 15°, and in the intrinsic equation is 60°.

15. Similarly, in (ii.),

$$s = \int \frac{\sqrt{(1+t^2)}\,dt}{\sqrt{(1-t^6)}}$$

$$= \int \frac{d\psi}{\sqrt{(\cos^6\psi - \sin^6\psi)}}$$

$$= \int \frac{d\psi}{\sqrt{\{\cos 2\psi\,(1-\frac{1}{4}\sin^2 2\psi)\}}};$$

and, generally, $$s = \int \frac{\sqrt{(1+t^2)}\,dt}{\sqrt{(\kappa t^6+\lambda)}} = \int \frac{d\psi}{\sqrt{(\kappa\cos^6\psi+\lambda\sin^6\psi)}},$$

which is expressible as the sum of two elliptic integrals of the first kind.

16. The Orthogonal Reciprocant

$$(t^2+1)\,c-10abt+15a^3=0,$$

obtained by integrating the above Mixed Reciprocant, has been integrated by Mr. Hammond (*Nature*, Jan. 7, 1886, p. 231) in the form

$$x = \int \frac{dt}{\sqrt{\{\kappa(1-15t^2+15t^4-t^6)+\lambda(6t-20t^3+6t^5)\}}}+\mu,$$

$$y = \int \frac{t\,dt}{\sqrt{\{\kappa(1-15t^2+15t^4-t^6)+\lambda(6t-20t^3+6t^5)\}}}+\nu;$$

changing his λ into 2λ; and then we see that

$$x+iy = \int \frac{(1+it)\,dt}{\sqrt{\{\frac{1}{2}(\kappa-i\lambda)(1+it)^6+\frac{1}{2}(\kappa+i\lambda)(1-it)^6\}}}+\mu+i\nu.$$

17. By a change of origin we can make μ and ν vanish, and by turning the axes through an angle $\frac{1}{6}\tan^{-1}\lambda/\kappa$ we can make λ vanish; so that

$$x+iy = \frac{1}{\sqrt{(\frac{1}{2}\kappa)}}\int \frac{\dfrac{dt}{(1+it)^2}}{\sqrt{\left\{\left(\dfrac{1-it}{1+it}\right)^6+1\right\}}}$$

$$= \frac{1}{\sqrt{(\frac{1}{2}\kappa)}}\int \frac{d\left(\dfrac{1-it}{1+it}\right)}{\sqrt{\left\{-\left(\dfrac{1-it}{1+it}\right)^6-1\right\}}}$$

$$= \frac{1}{\sqrt{(\frac{1}{2}\kappa)}}\int \frac{d\left(\dfrac{1+it}{1-it}\right)^2}{\sqrt{\left\{-4\left(\dfrac{1+it}{1-it}\right)^6-4\right\}}};$$

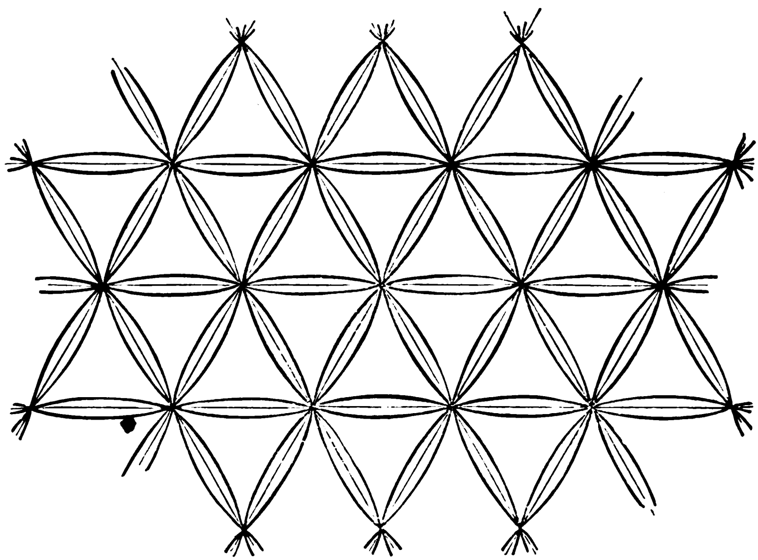
so that, replacing $\frac{1}{2}\kappa$ by unity, which may be done without loss of generality, $$\left(\frac{1+it}{1-it}\right)^2 = -\wp(x+iy;\,0,\,4).$$

18. Changing the sign of i,

$$\left(\frac{1-it}{1+it}\right)^2 = -\wp\,(x-iy\,;\ 0,\ 4),$$

so that
$$\wp\,(x+iy)\,\wp\,(x-iy) = 1\,;$$

agreeing with Mr. Rogers's form (*Proc. Lond. Math. Soc.*, March, 1886), and giving a curve as in the annexed figure.



Also, with $g_2 = 0,\ g_3 = 4$,

$$\wp\,(x+iy)\,\wp\,(x-iy) = \frac{\wp^2 x\,\wp^2 iy + 4\,(\wp x + \wp iy)}{(\wp x - \wp iy)^2},$$

and the numerical values of $\wp x$ and $\wp iy$ are given in the Table, calculated by Mr. Hadcock, in *Proc. Lond. Math. Soc.*, Vol. XVII., p. 268.

We may also write the above relations as

$$\cos 4\psi + i \sin 4\psi = -\wp\,(x+iy),$$

$$\text{co}\ \ 4\psi - i \sin 4\psi = -\wp\,(x-iy).$$

19. When $g_2 = 0$, and ω denotes an imaginary cube root of unity,

$$\wp \omega u = \omega \wp u,$$

the simplest case of *Complex Multiplication* of Weierstrass's Elliptic Functions; so that our equation above (§ 18) may be written

$$\wp \omega (x+iy)\, \wp \omega^2 (x-iy) = 1,$$

showing that the coordinate axes may be turned through 120° without alteration of appearance, as indicated in the figure.

In Legendre's and Jacobi's notation, the equation of the curve to a different scale, with $8\sqrt{3}\kappa = 1$, and $\lambda = 0$ in § 16, may be written

$$k'^2 \operatorname{tn}^2 (x, k) = k^2 \operatorname{tn}^2 (y, k')$$

for the inclined branches; and

$$k^2 \operatorname{sn}^2 (x, k) = k'^2 \operatorname{sn}^2 (y, k')$$

for the horizontal branches (Sylvester, *American Journal of Mathematics*, Vol. VIII., p. 235), with $k = \sin 15°$, $k' = \sin 75°$; equivalent

to $$\operatorname{am} (x \pm K,\ k) = \operatorname{am} (y \pm K', k'),$$

or $$\operatorname{am} (x \pm iK', k) = \operatorname{am} (y \pm iK,\ k'),$$

where $$K'/K = \sqrt{3}.$$

20. This Reciprocant

$$(1+t^2)\, c - 10abt + 15a^3 = 0,$$

when expressed in the intrinsic form, has been shown by Captain MacMahon to become $\dfrac{d^3\psi}{ds^3} + 18\left(\dfrac{d\psi}{ds}\right)^3 = 0.$

Integrating this equation,

$$\left(\frac{d^2\psi}{ds^2}\right)^2 = C - 9\left(\frac{d\psi}{ds}\right)^4,$$

or, putting $\dfrac{d\psi}{ds} = \dfrac{1}{\rho} = q$; and denoting by m the maximum value of q,

$$\left(\frac{dq}{ds}\right)^2 = 9\,(m^4 - q^4),$$

the solution of which is

$$\frac{d\psi}{ds} = q = m \operatorname{cn}\left(3\sqrt{2}\, ms,\ \frac{1}{\sqrt{2}}\right).$$

Therefore $$3\sqrt{2}\,\psi = \int \operatorname{cn}(3\sqrt{2}\, ms)\, d\,(3\sqrt{2}\, ms)$$
$$= \sqrt{2} \sin^{-1}\left\{\frac{1}{\sqrt{2}} \operatorname{sn}(3\sqrt{2}\, ms)\right\},$$

or
$$\operatorname{sn}(3\sqrt{2}\,ms) = \sqrt{2}\sin 3\psi,$$
$$\sin 3\psi = \frac{1}{\sqrt{2}}\operatorname{sn}(3\sqrt{2}\,ms),$$
or
$$\cos 3\psi = \operatorname{dn}(3\sqrt{2}\,ms).$$

This is the intrinsic equation of a curve whose equation in Cartesian coordinates, using Jacobi's elliptic functions, is of one of the preceding forms of § 19, or

$$\operatorname{dn}(x, k)\operatorname{dn}(y, k') = k,$$

where $k = \sin 15°$, $k' = \sin 75°$, as mentioned in Mr. Hammond's paper (*Proc. Lond. Math. Soc.*, Vol. XVII., p. 130); and then another curious result is obtained, analogous to that of § 14, of a curve, whose Cartesian equation involves elliptic functions of modular angle 15°, having its arc expressed by an elliptic integral of the first kind of modular angle 45°; no simple transformation existing from one modulus to the other.

III. *Euler's Equations of Motion.*

21. These well-known equations, when there are no impressed forces, written in the form

$$\left.\begin{aligned} A\frac{dp}{dt} - (B-C)\,qr &= 0 \\ B\frac{dq}{dt} - (C-A)\,rp &= 0 \\ C\frac{dr}{dt} - (A-B)\,pq &= 0 \end{aligned}\right\},$$

are satisfied by
$$\left.\begin{aligned} Ap^2 &= -(B-C)(z-e_1) \\ Bq^2 &= -(C-A)(z-e_2) \\ Cr^2 &= -(A-B)(z-e_3) \end{aligned}\right\},$$

provided that

$$\frac{dz}{dt} = 2pqr,$$

or
$$\frac{dz^2}{dt^2} = 4p^2q^2r^2$$
$$= -4\frac{(B-C)(C-A)(A-B)}{ABC}(z-e_1)(z-e_2)(z-e_3)$$
$$= -\frac{(B-C)(C-A)(A-B)}{ABC}(4z^3 - g_2z - g_3);$$

so that
$$z = \wp u,$$

where
$$\frac{du^2}{dt^2} = -\frac{(B-C)(C-A)(A-B)}{ABC} = M^2, \text{ suppose,}$$

since $(B-C)(C-A)(A-B)$ is negative; and then
$$u = Mt + \text{a constant.}$$

22. Then
$$T = Ap^2 + Bq^2 + Cr^2$$
$$= \quad (B-C)\,e_1 + \quad (C-A)\,e_2 + \quad (A-B)\,e_3;$$

and
$$G^2 = A^2p^2 + B^2q^2 + C^2r^2$$
$$= A\,(B-C)\,e_1 + B\,(C-A)\,e_2 + C\,(A-B)\,e_3;$$

also
$$0 = \qquad e_1 + \qquad e_2 + \qquad e_3;$$

so that
$$e_1 = \frac{G^2\,(-2A+B+C) - T\,(2BC-CA-AB)}{3\,(B-C)(C-A)(A-B)},$$

$$e_2 = \frac{G^2\,(A-2B+C) - T\,(-BC+2CA-AB)}{3\,(B-C)(C-A)(A-B)},$$

$$e_3 = \frac{G^2\,(A+B-2C) - T\,(-BC-CA+2AB)}{3\,(B-C)(C-A)(A-B)};$$

and then
$$g_2 = -4\,(e_2e_3 + e_3e_1 + e_1e_2), \quad g_3 = 4e_1e_2e_3.$$

Also
$$e_2 - e_3 = \frac{AT - G^2}{(C-A)(A-B)},$$

$$e_3 - e_1 = \frac{BT - G^2}{(A-B)(B-C)},$$

$$e_1 - e_2 = \frac{CT - G^2}{(B-C)(C-A)};$$

and
$$g_2 = \tfrac{2}{3}\left\{(e_2-e_3)^2 + (e_3-e_1)^2 + (e_1-e_2)^2\right\}.$$

Also the *discriminant*
$$D = g_2^3 - 27g_3^2$$
$$= 16\,(e_2-e_3)^2\,(e_3-e_1)^2\,(e_1-e_2)^2$$
$$= 16\,\frac{(AT-G^2)^2\,(BT-G^2)^2\,(CT-G^2)^2}{(B-C)^4\,(C-A)^4\,(A-B)^4}.$$

23. Supposing $A > B > C$; then

(i.) When the polhode encloses the axis A, $BT - G^2$ is negative, and then $e_2 - e_3$ is negative, $e_3 - e_1$ is negative, and $e_1 - e_2$ is positive; so that $e_1 > e_3 > e_2$, and $\wp u$ oscillates in value between e_2 and e_3, so that we have
$$u = Mt + \omega_3;$$

(ii.) when the polhode encloses the axis C, $BT - G^2$ is positive, and then $e_2 - e_3$ is negative, $e_3 - e_1$ is positive, and $e_1 - e_2$ is positive; so that $e_3 > e_1 > e_2$, and $\wp u$ oscillates in value between e_1 and e_2, so that
$$u = Mt + \omega_2,$$
as before.

IV. *The Spherical Pendulum and Top.*

24. In the spherical pendulum, of length l, the equations of motion may be written
$$\tfrac{1}{2} l^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\psi}^2) = g(b - l\cos\theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
the *equation of energy;* and
$$l^2 \sin^2\theta\, \dot{\psi} = G \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$
the *equation of conservation of angular momentum* about the vertical; using as coordinates, θ the polar distance on the sphere in circular measure from the highest point, and ψ the longitude.

Eliminating $\dot{\psi}$ between (1) and (2),
$$\tfrac{1}{2} l^2 \sin^2\theta\, \dot{\theta}^2 + \tfrac{1}{2}\frac{G^2}{l^2} = g(b - l\cos\theta)(1 - \cos^2\theta),$$
or
$$\tfrac{1}{2}\sin^2\theta\, \dot{\theta}^2 = \frac{g}{l}(\cos^2\theta - 1)\left(\cos\theta - \frac{b}{l}\right) - \tfrac{1}{2}\frac{G^2}{l^4} \quad \ldots\ldots\ldots (3).$$

25. Put $\cos\theta = 2z + \gamma$; then
$$2\dot{z}^2 = \frac{g}{l}\left\{(4z^2 + 4\gamma z + \gamma^2 - 1)\left(2z + \gamma - \frac{b}{l}\right) - \tfrac{1}{2}\frac{G^2}{gl^3}\right\}$$
$$= \frac{g}{l}\left\{8z^3 + 4\left(3\gamma - \frac{b}{l}\right)z^2 + \left(6\gamma^2 - 4\gamma\frac{b}{l} - 2\right)z \right.$$
$$\left. + (\gamma^2 - 1)\left(\gamma - \frac{b}{l}\right) - \tfrac{1}{2}\frac{G^2}{gl^3}\right\};$$
and then, if
$$3\gamma = b/l,$$
$$\dot{z}^2 = \frac{g}{l}\left\{4z^3 - (3\gamma^2 + 1)z - \gamma(\gamma^2 - 1) - \frac{G^2}{4gl^3}\right\},$$

so that $$z = \wp(u;\, g_2, g_3),$$

and $$u = \sqrt{\frac{g}{l}}\, t + \text{a constant},$$

where $$g_2 = 3\gamma^2+1, \quad g_3 = \gamma(\gamma^2-1) - G^2/4gl^3;$$

and $$\cos\theta = 2\wp u + \gamma,$$

where $$\gamma = \tfrac{1}{3}b/l.$$

26. If $G = 0$, then the *discriminant*

$$D = (1-9\gamma^2)^2 \text{ (Salmon, \textit{Higher Algebra}, p. 171)},$$

and the solution of the simple circular pendulum is obtained.

Then (i.) when the pendulum oscillates, $\cos\theta$ ranges from -1 to b/l, and $\wp u$ ranges from e_3 to e_2, where

$$e_1 = -\tfrac{1}{2}(\gamma-1), \quad e_2 = \gamma, \quad e_3 = -\tfrac{1}{2}(\gamma+1),$$

and therefore $$u = \sqrt{\frac{g}{l}}\, t + \omega_3.$$

In small oscillations, $b = -l$, $\gamma = -\tfrac{1}{3}$, and $e_2 = e_3$.

(ii.) When the pendulum revolves, $\cos\theta$ ranges from -1 to $+1$, and $\wp u$ ranges from e_3 to e_2, where

$$e_1 = \gamma, \quad e_2 = -\tfrac{1}{2}(\gamma-1), \quad e_3 = -\tfrac{1}{2}(\gamma+1),$$

and therefore $$u = \sqrt{\frac{g}{l}}\, t + \omega_3,$$

as before.

In the separating case, $b = l$, $\gamma = \tfrac{1}{3}$, and $e_1 = e_2$.

27. Returning to the spherical pendulum,

$$\begin{aligned} l^2\dot{\psi} &= \frac{G}{\sin^2\theta} \\ &= \tfrac{1}{2}\frac{G}{1-\cos\theta} + \tfrac{1}{2}\frac{G}{1+\cos\theta} \\ &= \tfrac{1}{4}\frac{G}{-\tfrac{1}{2}(\gamma-1)-\wp u} + \tfrac{1}{4}\frac{G}{\wp u + \tfrac{1}{2}(\gamma+1)} \\ &= \tfrac{1}{4}\frac{G}{\wp a - \wp u} + \tfrac{1}{4}\frac{G}{\wp u - \wp b}, \end{aligned}$$

putting $$\wp a = -\tfrac{1}{2}(\gamma-1), \quad \wp b = -\tfrac{1}{2}(\gamma+1).$$

Then $\sin^2 \frac{1}{2}\theta = \wp a - \wp u, \quad \cos^2 \frac{1}{2}\theta = \wp u - \wp b,$

since $$\wp a - \wp b = 1,$$

and $$\tan^2 \tfrac{1}{2}\theta = \frac{\wp a - \wp u}{\wp u - \wp b};$$

also $$\wp'^2 a = \wp'^2 b = -G^2/4gl^3;$$

so that $$\frac{d\psi}{du} = \tfrac{1}{2}\frac{i\wp' a}{\wp u - \wp a} + \tfrac{1}{2}\frac{i\wp' b}{\wp u - \wp b},$$

since, as we shall see in § 34, $\wp' a$ is positive imaginary, and $\wp' b$ is negative imaginary (Maggi, *Rendiconti, Reale Istituto Lombardo*, Serie II., Vol. XVII., Pisa, 1884).

28. In this case of the spherical pendulum, a and b are connected by the relation $$\wp'^2 a = \wp'^2 b,$$

so that $$\wp' a = -\wp' b,$$

equivalent to $$\wp(a-b) + \wp a + \wp b = 0,$$

an equation discussed by Halphen in the *Journal de l'Ecole Polytechnique*, 54 Cahier, 1884: *Note sur l'Inversion des Intégrales Elliptiques.*

It may be noticed here that the solution of this equation, when the invariant $g_2 = 0$, is $a = \omega b$, where ω denotes a real or imaginary cube or sixth root of unity.

29. In the more general case of the motion of the Top, or solid of revolution, moving under gravity about a fixed point in its axis, the previous equations of motion for the spherical pendulum are but slightly modified; equation (1) being again applicable, and equation (2) must be changed to

$$A \sin^2\theta\,\dot{\psi} + Cn\cos\theta = G \quad \text{.......................(3)}$$

(*Quarterly Journal of Mathematics*, Vol. XV.: "On the Motion of a Top, and Allied Problems in Dynamics").

Again, eliminating $\dot{\psi}$ as before,

$$\tfrac{1}{2}\sin^2\theta\,\dot{\theta}^2 = \frac{g}{l}(\cos^2\theta - 1)\left(\cos\theta - \frac{b}{l}\right) - \tfrac{1}{2}\left(\frac{G - Cn\cos\theta}{A}\right)^2$$

$$= \frac{g}{l}(\cos\theta - d)(\cos\theta - \cos\alpha)(\cos\theta - \cos\beta),$$

suppose, θ being supposed to lie between α and β, so that $\alpha < \theta < \beta$.

30. Putting, as before, $\cos\theta = 2z+\gamma$,

then $$\dot{z}^2 = \frac{g}{l}(4z^3-g_2 z-g_3),$$

provided that $$3\gamma = \frac{b}{l}+\tfrac{1}{2}\frac{l}{g}\frac{C^2n^2}{A^2};$$

so that $$z = \wp u,$$

where $$u = \sqrt{\frac{g}{l}}\,t + \text{a constant},$$

and $$g_2 = 3\gamma^2+1-GCnl/gA^2,$$
$$g_3 = \gamma(\gamma^2-1)+(G^2-2GCn\gamma+C^2n^2)\,l/4gA^2.$$

31. Then if, as before, for the spherical pendulum,

$$u = a \text{ when } \cos\theta = 1, \quad \text{so that } \wp a = -\tfrac{1}{2}(\gamma-1),$$
$$u = b \quad ,, \quad \cos\theta = -1, \quad ,, \quad \wp b = -\tfrac{1}{2}(\gamma+1),$$

then $$\sin^2\tfrac{1}{2}\theta = \wp a-\wp u, \quad \cos^2\tfrac{1}{2}\theta = \wp u-\wp b,$$

and $$\wp'^2 a = -\tfrac{1}{4}\frac{l}{g}\left(\frac{G-Cn}{A}\right)^2,$$
$$\wp'^2 b = -\tfrac{1}{4}\frac{l}{g}\left(\frac{G+Cn}{A}\right)^2.$$

From $$\frac{d\psi}{dt} = \frac{G-Cn\cos\theta}{A\sin^2\theta}$$
$$= \tfrac{1}{2}\frac{G-Cn}{A}\frac{1}{1-\cos\theta}+\tfrac{1}{2}\frac{G+Cn}{A}\frac{1}{1+\cos\theta}$$

we obtain, as before,

$$\frac{d\psi}{du} = \tfrac{1}{2}\frac{i\wp' a}{\wp u-\wp a}+\tfrac{1}{2}\frac{i\wp' b}{\wp u-\wp b} \quad \text{......................}(4).$$

32. Introducing at this stage σu, the *sigma function* of Weierstrass,

as defined by $$\frac{d^2}{du^2}\log\sigma u = -\wp u,$$

and also the fundamental formula

$$\wp u-\wp v = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v},$$

called by Schwarz the *pocket edition* of the elliptic functions; differentiating this formula logarithmically with respect to u and v,

$$\frac{\wp' u}{\wp u - \wp v} = \frac{\sigma'(u+v)}{\sigma(u+v)} + \frac{\sigma'(u-v)}{\sigma(u-v)} - 2\frac{\sigma' u}{\sigma u},$$

$$\frac{-\wp' v}{\wp u - \wp v} = \frac{\sigma'(u+v)}{\sigma(u+v)} - \frac{\sigma'(u-v)}{\sigma(u-v)} - 2\frac{\sigma' v}{\sigma v};$$

and integrating with respect to v and u, respectively,

$$\int \frac{\wp' u\, dv}{\wp u - \wp v} = \log \frac{\sigma(u+v)}{\sigma(u-v)} - 2v\frac{\sigma' u}{\sigma u},$$

$$\int \frac{\wp' v\, du}{\wp u - \wp v} = \log \frac{\sigma(u-v)}{\sigma(u+v)} + 2u\frac{\sigma' v}{\sigma v};$$

Weierstrass's form of the Third Elliptic Integral; so that

$$\int \frac{\wp' u\, dv + \wp' v\, du}{\wp u - \wp v} = 2u\frac{\sigma' v}{\sigma v} - 2v\frac{\sigma' u}{\sigma u},$$

corresponding to Jacobi's formula for the interchange of *argument* and *parameter* in the third elliptic integral.

33. Then equation (4) becomes

$$\begin{aligned}\frac{d\psi}{du} = {} & \tfrac{1}{2}i\frac{\sigma'(u-a)}{\sigma(u-a)} - \tfrac{1}{2}i\frac{\sigma'(u+a)}{\sigma(u+a)} + i\frac{\sigma' a}{\sigma a} \\ & + \tfrac{1}{2}i\frac{\sigma'(u-b)}{\sigma(u-b)} - \tfrac{1}{2}i\frac{\sigma'(u+b)}{\sigma(u+b)} + i\frac{\sigma' b}{\sigma b};\end{aligned}$$

and, integrating,

$$\psi = \tfrac{1}{2}i \log \frac{\sigma(u-a)\,\sigma(u-b)}{\sigma(u+a)\,\sigma(u+b)} + i\left(\frac{\sigma' a}{\sigma a} + \frac{\sigma' b}{\sigma b}\right) u \quad \ldots\ldots\ldots (5).$$

34. The values ω_1, ω_2, ω_3 of u, and therefore the values e_1, e_2, e_3 of $\wp u$, correspond to the values d, $\cos\alpha$, $\cos\beta$ of $\cos\theta$; so that

$$\frac{1-d}{1+d} = \frac{\wp a - e_1}{e_1 - \wp b} = -\left(\frac{\sigma_1 a}{\sigma\, a}\right)^2 \Big/ \left(\frac{\sigma_1 b}{\sigma\, b}\right)^2,$$

$$\frac{1-\cos\alpha}{1+\cos\alpha} = \tan^2 \tfrac{1}{2}\alpha = -\left(\frac{\sigma_2 a}{\sigma\, a}\right)^2 \Big/ \left(\frac{\sigma_2 b}{\sigma\, b}\right)^2,$$

$$\frac{1-\cos\beta}{1+\cos\beta} = \tan^2 \tfrac{1}{2}\beta = -\left(\frac{\sigma_3 a}{\sigma\, a}\right)^2 \Big/ \left(\frac{\sigma_3 b}{\sigma\, b}\right)^2.$$

In order that $\wp u$ should oscillate in magnitude between e_2 and e_3,

we must put
$$u = \sqrt{\frac{g}{l}}\,t + \omega_3.$$

Also, since
$$e_1 > \wp a > e_2,$$

therefore we can put $a = \omega_1 + r\omega_3$, when r is a proper fraction; and then $\wp' a$ is positive imaginary; and, since

$$e_3 > \wp b > -\infty,$$

therefore we can put $b = s\omega_3$, where s is a proper fraction; and then $\wp' b$ is negative imaginary.

35. When $G = 0$ or $Cn = 0$, then $\wp'^2 a = \wp'^2 b$, and the motion of the Top is directly comparable with that of a spherical pendulum.

When G and Cn are both zero, the Top oscillates in a vertical plane like a simple circular pendulum, and then $a = \omega_1$, $b = \omega_3$.

If $a = \omega_1$, then
$$G - Cn = 0;$$

and if $b = \omega_3$, then
$$G + Cn = 0.$$

If $a + b = \omega_1 + \omega_3$, then
$$G - Cn \cos\alpha = 0,$$

and the trace of the axis on the unit sphere of reference has a series of cusps on the parallel of latitude $\theta = \alpha$.

36. According to the method of Hermite (*Sur quelques Applications des Fonctions Elliptiques*, 1885, p. 109), taking the axis of z vertically upwards, the equations of motion of the spherical pendulum are

$$\frac{d^2x}{dt^2} + N\frac{x}{l} = 0,$$

$$\frac{d^2y}{dt^2} + N\frac{y}{l} = 0,$$

$$\frac{d^2z}{dt^2} + N\frac{z}{l} = -g;$$

with
$$x^2 + y^2 + z^2 = l^2;$$

the two first integrals of which are

$$\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = g(c - z),$$
$$x\dot{y} - \dot{x}y = G.$$

Then, as before,
$$z = l\cos\theta = l(2\wp u + \gamma),$$

where
$$\gamma = \tfrac{1}{3}c/l.$$

Also $$x\ddot{x}+y\ddot{y}+z\ddot{z}+Nl=-gz,$$

or, since $$x\ddot{x}+y\ddot{y}+z\ddot{z}+\dot{x}^2+\dot{y}^2+\dot{z}^2=0,$$

$$\begin{aligned}Nl&=-gz+\dot{x}^2+\dot{y}^2+\dot{z}^2\\&=-gz+2g(c-z)=g(2c-3z),\end{aligned}$$

N representing the pressure per unit mass on the sphere.

37. Then $$\frac{d^2}{dt^2}(x+iy)=-\frac{N}{l}(x+iy)=\frac{g}{l^2}(3z-2c)(x+iy),$$

or $$\frac{d^2}{du^2}(x+iy)=(6\wp u-3\gamma)(x+iy),$$

Lamé's differential equation for $n=2$; and the solution is

$$x+iy=2il\frac{\sigma(u+a)\,\sigma(u+b)}{\sigma a\,\sigma b\,\sigma^2 u}\exp\left(-\frac{\sigma' a}{\sigma a}-\frac{\sigma' b}{\sigma b}\right)u,$$

or $$x-iy=2il\frac{\sigma(u-a)\,\sigma(u-b)}{\sigma a\,\sigma b\,\sigma^2 u}\exp\left(\frac{\sigma' a}{\sigma a}+\frac{\sigma' b}{\sigma b}\right)u,$$

where $$\gamma=\wp(a-b)=-\wp a-\wp b.$$

This may also be obtained by combining the value of $e^{i\psi}$ from equation (5) with the result of § 27 or § 31,

$$\sin^2\theta=4(\wp a-\wp u)(\wp u-\wp b);$$

and it is interesting to compare this result with Hermite's (p. 112)

$$x+iy=AD_u\frac{H'OH(u+\omega)}{\Theta\omega\,\Theta u}\exp\left(\lambda-\frac{\Theta'\omega}{\Theta\omega}\right)u,$$

where ω and λ are constants; the equivalence of the two forms being secured by putting $\omega=a+b$, and

$$\lambda=\wp'(a-b)=\wp' a=-\wp' b=\zeta(a+b)-\zeta a-\zeta b,$$

using Halphen's notation (Chapter v.) ζa for $\dfrac{\sigma' a}{\sigma a}$; also

$$\zeta(a-b)=\zeta a-\zeta b.$$

V. *The Trajectory for the Cubic Law of Resistance.*

38. In Volume XIV. of the *Proceedings of the Royal Artillery Institution* the trajectory of a projectile in a resisting medium, with a tangential resistance varying as the cube of the velocity, is investigated, and it is there shown that a great simplification is effected by the employment of Weierstrass's functions.

The equation of the trajectory referred to oblique axes, one the

tangent at the point of infinite velocity, and the other vertical, is then

$$y = -3x\,\zeta b - \log\sigma(b-x) - \omega^2\log\sigma(b-\omega x) - \omega\,\log\sigma(b-\omega^2 x)\,;$$

and the time of flight is given by

$$t = - \qquad \log\sigma(b-x) - \omega\,\log\sigma(b-\omega x) - \omega^2\log\sigma(b-\omega^2 x)\,;$$

b denoting the value of x at the vertical asymptote.

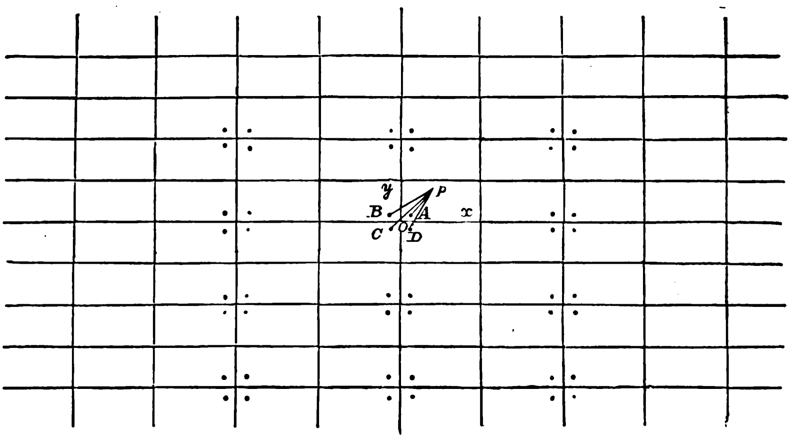
VI. *Uniplanar Electrical and Hydrodynamical Problems.*

39. Referring to the *Quarterly Journal of Mathematics*, Vols. XVII. and XVIII., "Solution by means of Elliptic Functions of some Problems in the Conduction of Heat and of Electricity," and "Functional Images in Cartesians," for the statement of the problems to be solved and of the notation employed; then, for a source of strength 2π at $z' = x' + iy'$, within the rectangle bounded by $x = 0$, $x = a$, $y = 0$, $y = b$,

$$\phi + i\psi = \log \sigma\alpha\ \sigma\beta\ \sigma\gamma\ \sigma\delta,$$

and for a vortex of circulation 2π at z',

$$\phi_1 + i\psi_1 = \log\frac{\sigma\alpha\ \sigma\gamma}{\sigma\beta\ \sigma\delta}.$$



Here

$$\alpha = x - x' + i\,.\,y - y',$$
$$\beta = x + x' + i\,.\,y - y',$$
$$\gamma = x + x' + i\,.\,y + y',$$
$$\delta = x - x' + i\,.\,y + y'\,;$$

so that α, β, γ, δ represent the four vectors AP, BP, CP, DP, proceeding to any point P from A at z', and the images B, C, D of A in the coordinate axes.

The remaining images form similar groups of four, round centres in the plane whose coordinates are $2ma$, $2m'b$, where m and m' are integers.

To be accurate, the sigma functions should have certain simple exponential factors, but these are cancelled by placing an equal and opposite source, *i.e.* a *sink*, inside the rectangle, and then the physical impossibility of zero flow across a boundary with a single source is removed; and by placing the source and sink at corners of the rectangle, we obtain the various results of the article in the *Quarterly Journal*, XVII., "Solution by means of Elliptic Functions, &c."; and now $\omega_1 = a$, $\omega_3 = ib$.

40. Transforming the coordinates by the use of conjugate functions, given by
$$x+iy = \mathrm{f}(\chi+i\rho),$$
then, for a right-angled quadrilateral figure bounded by
$$\chi = \chi_0, \quad \chi = \chi_1, \quad \rho = \rho_0, \quad \rho = \rho_1,$$
we must put the vectors
$$\begin{aligned} \alpha &= \chi-\chi' \qquad +i\,.\,\rho-\rho', \\ \beta &= \chi+\chi'-2\chi_0+i\,.\,\rho-\rho', \\ \gamma &= \chi+\chi'-2\chi_0+i\,.\,\rho+\rho'-2\rho_0, \\ \delta &= \chi-\chi' \qquad +i\,.\,\rho+\rho'-2\rho_0. \end{aligned}$$
Then, for a source at (χ', ρ'),
$$\phi+i\psi = \log \sigma\alpha \;\; \sigma\beta \;\; \sigma\gamma \;\; \sigma\delta;$$
and for a vortex or an electrified point,
$$\phi_1+i\psi_1 = \log \frac{\sigma\alpha \;\; \sigma\gamma}{\sigma\beta \;\; \sigma\delta};$$
and here
$$\omega_1 = \chi_1-\chi_0, \quad \omega_3 = i\,.\,\rho_1-\rho_0,$$
the periods of the Weierstrass functions.

41. For a doubly connected plane region, bounded by $\rho=\rho_0$ and $\rho=\rho_1$, we may put, as in "Functional Images in Cartesians,"
$$\phi+i\psi = \log \sigma(\chi-\chi'+i\,.\,\rho-\rho')\,\sigma(\chi-\chi'+i\,.\,\rho+\rho'-2\rho_0),$$
$$\phi_1+i\psi_1 = \log \frac{\sigma(\chi-\chi'+i\,.\,\rho-\rho')}{\sigma(\chi-\chi'+i\,.\,\rho+\rho'-2\rho_0)},$$
supposing χ to increase by ω_1 in a complete circuit of the region.

It is easily verified in these expressions, from the formulæ given by

Schwarz, that ψ and ϕ_1 have constant values round the boundaries $\chi = \chi_0$, $\chi = \chi_1$, $\rho = \rho_0$, $\rho = \rho_1$, or can be made constant by the addition of simple expressions.

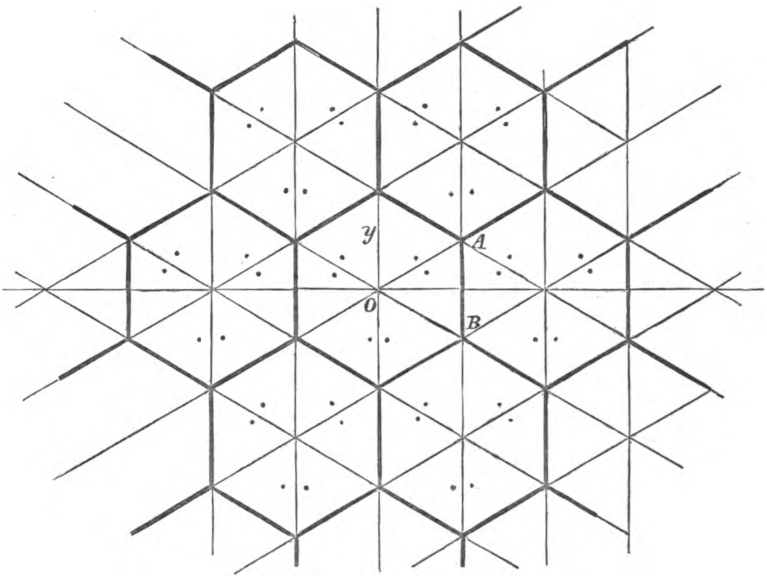
42. When a source or vortex is placed at z' inside an equilateral triangle OAB, then the vectors of the images are given by

$$\omega z', \ \omega^2 z', \text{ and } z'', \ \omega z'', \ \omega^2 z'',$$

where $z'' = -x' + iy'$, and $\omega^3 = 1$, so that ω denotes an imaginary cube root of unity; and similar groups of six images ranged round centres of hexagons forming a tesselated pavement, the coordinates of the centres being

$$2mh, \ 2m'h\sqrt{3}, \text{ and } (2m+1)\,h, \ (2m'+1)\,h\sqrt{3},$$

where h denotes the altitude of the equilateral triangle, and m and m' are integers.



Then, for a source inside an equilateral triangle, like OAB,

$$\begin{aligned}\phi + i\psi = \log \sigma\,(z-z')\,\sigma\,(z-\omega z')\,\sigma\,(z-\omega^2 z')\\ \sigma_2\,(z-z')\,\sigma_2\,(z-\omega z')\,\sigma_2\,(z-\omega^2 z')\\ \sigma\,(z-z'')\,\sigma\,(z-\omega z'')\,\sigma\,(z-\omega^2 z'')\\ \sigma_2\,(z-z'')\,\sigma_2\,(z-\omega z'')\,\sigma_2\,(z-\omega^2 z'');\end{aligned}$$

and for a vortex, or electrified point,

$$\phi_1 + i\psi_1 = \log \frac{\sigma\,(z-z')\;\sigma\,(z-\omega z')\;\sigma\,(z-\omega^2 z')}{\sigma\,(z-z'')\;\sigma\,(z-\omega z'')\;\sigma\,(z-\omega^2 z'')}$$

$$\frac{\sigma_2\,(z-z')\;\sigma_2\,(z-\omega z')\;\sigma_2\,(z-\omega^2 z')}{\sigma_2\,(z-z'')\;\sigma_2\,(z-\omega z'')\;\sigma_2\,(z-\omega^2 z'')};$$

and here $$\omega_1 = h, \quad \omega_3 = ih\sqrt{3},$$

so that $$\frac{\omega_3}{\omega_1} = i\frac{K'}{K} = i\sqrt{3},$$

and therefore the modular angle is 15°.

(O. Zimmerman, *Das Logarithmische Potential einer gleichseitig dreieckigen Platte*, Diss. Jena. 1880.)

VII. *Attractions.*

43. The well-known expression for the potential V of the homogeneous ellipsoid $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

of mass M at an external point x, y, z, viz.,

$$V = \tfrac{3}{4}M\int_\lambda^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda\,.\,b^2+\lambda\,.\,c^2+\lambda)}}\left(1 - \frac{x^2}{a^2+\lambda} - \frac{y^2}{b^2+\lambda} - \frac{z^2}{c^2+\lambda}\right),$$

where $a^2+\lambda$, $b^2+\lambda$, $c^2+\lambda$ are the squares of the semi-axes of the confocal ellipsoid passing through the point x, y, z, is reduced to Weierstrass's functions by putting

$$a^2+\lambda = \wp u - e_1, \quad b^2+\lambda = \wp u - e_2, \quad c^2+\lambda = \wp u - e_3,$$

supposing $a^2 < b^2 < c^2$, and therefore $e_1 > e_2 > e_3$.

Then $$\wp u = \tfrac{1}{3}(a^2+b^2+c^2) + \lambda,$$

since $$e_1 + e_2 + e_3 = 0;$$

and $e_1 = \frac{1}{3}(-2a^2+b^2+c^2)$, $e_2 = \frac{1}{3}(a^2-2b^2+c^2)$, $e_3 = \frac{1}{3}(a^2+b^2-2c^2)$;

so that $$g_2 = -4\,(e_2e_3 + e_3e_1 + e_1e_2) = 2\,(e_1^2+e_2^2+e_3^2)$$
$$= \tfrac{2}{3}\left\{(b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2\right\};$$

and $$D = g_2^3 - 27g_3^2 = (e_2-e_3)^2\,(e_3-e_1)^2\,(e_1-e_2)^2$$
$$= (b^2-c^2)^2\,(c^2-a^2)^2\,(a^2-b^2)^2.$$

Then $$\int_\lambda^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda\,.\,b^2+\lambda\,.\,c^2+\lambda)}} = 2u,$$

and $$\frac{V}{\frac{3}{2}M} = \int_0^u \left(1 - \frac{x^2}{\wp u - e_1} - \frac{y^2}{\wp u - e_2} - \frac{z^2}{\wp u - e_3}\right) du;$$

also $$\int_0^u \frac{du}{\wp u - e_\lambda} = \frac{\zeta_\lambda u + e_\lambda u}{(e_\lambda - e_\mu)(e_\nu - e_\lambda)};$$

so that, if $a^2+\mu = \wp v - e_1$, $b^2+\mu = \wp v - e_2$, $c^2+\mu = \wp v - e_3$ are the squares of the semi-axes of the confocal hyperboloid of one sheet, and if

$$a^2+\nu = \wp w - e_1, \quad b^2+\nu = \wp w - e_2, \quad c^2+\nu = \wp w - e_3,$$

of the confocal hyperboloid of two sheets through the point (xyz), then

$$u = r\omega_1, \quad v = \omega_1 + s\omega_3, \quad w = t\omega_1 + \omega_3,$$

where r, s, t are proper fractions; and

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)} = \frac{\left(\frac{\sigma_1 u}{\sigma u}\right)^2 \left(\frac{\sigma_1 v}{\sigma v}\right)^2 \left(\frac{\sigma_1 w}{\sigma w}\right)^2}{(e_1-e_2)(e_1-e_3)},$$

with similar symmetrical expressions for y^2 and z^2.

On the Converse of Stereographic Projection and on Contangential and Coaxal Spherical Circles. By Mr. H. M. Jeffery, F.R.S.

[*Read May* 13*th*, 1886.]

On Systems of Spherical Circles.

1. The first section is on a form of conical projection and introduces the equations and processes herein used. The second treats of systems of coaxal and contangential circles. Next, similitude and inversion are defined and illustrated. Lastly, the processes are applied to the solution of the problem of Contacts.

In developing the analogies to Plane Geometry, it is shown that theorems which are distinct in Planimetry are dual in Spherics; that those which relate to the magnitude of angles are identical in both Geometries; while theorems on arcs are modified when the radius of the sphere becomes infinite.

On the Converse of Stereographic Projection.

2. By stereographic projection, curves on a sphere are projected on an equatorial plane, whose pole is the pole of projection. The converse process is here considered; lines and curves on the equatorial

or primitive plane are conically projected on the sphere from the same pole.

The two main familiar properties of this projection form the basis of the present investigation, (1) that all circles on the sphere are projected into circles or straight lines, constituting subcontrary sections of the cone of projection; (2) that an angle is unaltered by projection.

Spherical inversion and similitude, so far as spherical circles are concerned, are necessarily defined from this converse projection. (§ 21.) Since, in planimetry, an angle is unaltered by inversion, so in spherics it follows that an angle is also unaltered by spherical inversion, as is proved subsequently in § 24.

3. To obtain the formulæ of transformation by the converse of stereographic projection. (Fig. 1.)

Let R $(=1)$ be the radius of the sphere, of which $P'QP$ is a plane section, and E the centre. $ED = r$: $P'R$, its shadow on the sphere, $= \rho$; their mutual relation is

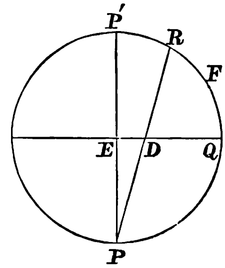


FIG. 1.

$$r = \tan\frac{\rho}{2}.$$

For $$\frac{r}{R} = \tan EPD = \tan\frac{\rho}{2} = \frac{\tan\rho}{1+\sec\rho}.$$

Also $$\frac{r^2}{R^2} = \tan^2\frac{\rho}{2} = \frac{\sec\rho - 1}{\sec\rho + 1}.$$

The Cartesian coordinates X $(= r\cos\theta)$, Y $(= r\sin\theta)$ are thus quadrically transformed into

$$\frac{\tan\rho\cos\theta}{1+\sec\rho},\quad \frac{\tan\rho\sin\theta}{1+\sec\rho},$$

or, in Gudermann's coordinate system and nomenclature,

$$\frac{x}{1+\sqrt{(x^2+y^2+1)}},\quad \frac{y}{1+\sqrt{(x^2+y^2+1)}}.$$

By x, y, ρ he denotes $\tan x$, $\tan y$, $\tan\rho$, for brevity; thus,

$$x = \tan\rho\cos\theta,\quad y = \tan\rho\sin\theta,\quad x^2+y^2+1 = \sec^2\rho.$$

4. To find the equation to a spherical circle referred to two tangent arcs as the axes of coordinates.

In plano, the equation to a circle referred to its two equal tangents (c) as axes is $\quad r^2 = x^2+y^2+2xy\cos\omega = 2c(x+y)-c^2$;

when projected conically, as in § 3, this becomes

$$\frac{\sec\rho-1}{\sec\rho+1}-\frac{2\tan t}{\sec t+1}\cdot\frac{x+y}{\sec\rho+1}+\frac{\sec t-1}{\sec t+1}=0;$$

when simplified, $\quad \sec\rho = \cos t+(x+y)\sin t \quad \text{.....................(A)},$

where $\quad \sec^2\rho = 1+x^2+y^2+2xy\cos\omega,$

and t is the tangent from O the origin. This equation may be deduced from the general form referred to oblique axes (Gudermann's *Sphärik*, § 6), $\quad \sec\rho = \cos t\,(1+Ax+By)$.

By x, y are denoted the tangents of the arcs intercepted on the quadrantal oblique axes AO, BO, by circles drawn through any point (P) from the extremities A, B.

When $x=0$, $\rho=t=y$; when $y=0$, $\rho=t=x$; hence $A=B=\tan t$. This form (A) is useful for studying the properties of contangential circles.

5. If a secant drawn from a point meet a circle in two points, whose distances are ρ_1, ρ_2, and the polar of that point at a distance R,

$$\cot\rho_1+\cot\rho_2 = 2\cot R.$$

The equation (A) may be written in the form

$$\left\{\sin t-(x+y)\cos t\right\}^2 = 2xy\,(1-\cos\omega).$$

As *in plano*, so in spherics thence derived by gnomonic projection,

$$x\operatorname{cosec}\beta = y\operatorname{cosec}\alpha = \tan\rho\operatorname{cosec}\omega;$$

if α, β denote the inclinations of the vector to the coordinate axes,

$$\cot\rho_1+\cot\rho_2 = 2\operatorname{cosec}\omega\cot t\,(\sin\alpha+\sin\beta) = 2\cot R.$$

Cor.—By modifying the equation so as to include all sphero-conics,

$$\left\{\sin t-(x+y)\cos t\right\}^2 = mxy$$

is applicable to them all, and is, in fact, the simplest case of Cotes' theorem adapted to Spherics.

6. The tangential biangular equation to a circle, plane or spherical, referred to the ends of a chord (or radical axis) as poles, is the dual of the preceding form (A),

$$\sec\rho = \cos c + (x+y)\sin c.$$

In this coordinate system (Fig. 2) (explained *Quarterly Math. Journal*, Vol. XIII., p. 130), a moveable tangent line (or, in Spherics, a great circle) $TCPD$ meets two fixed lines (or great circles) perpendicular to AB through its poles,

$$\rho = DTA, \quad c = EAB = FBA,$$

$$x = \tan DAB, \quad y = \tan CBA.$$

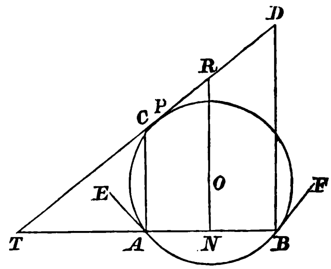


FIG 2.

In planimetry,

$$\cos c + (x+y)\sin c = -\frac{ON}{r} + \frac{AB\,(AC+BD)}{2r\times AB}$$

$$= \frac{RO}{r} = \operatorname{cosec} TRN = \sec\rho.$$

In spherics,

$$\cos c + (x+y)\sin c = -\frac{\sin ON}{\sin r} + \frac{\tan AN\,(\tan AC + \tan BD)}{\tan r\,\sin AB}.$$

The second term

$$= \tan\rho \operatorname{cosec} AB \tan AN \cot r\,(\sin AT + \sin BT)$$

$$= \tan\rho\cot r \sin TN \sec AN = \operatorname{cosec} r \tan RN \cos ON.$$

Hence $\quad \cos c + (x+y)\sin c = \operatorname{cosec} r \sin RO \sec RN$

$$= \operatorname{cosec} TRN \sec RN = \sec\rho.$$

This equation is useful for coaxal circles.

7. Dual theorem of § 5. If from a point in a given line three other lines be drawn, two of them touching a conic, and the third through the pole of the given line with respect to the conic,

$$\cot\rho_1 + \cot\rho_2 = 2\cot R,$$

if ρ_1, ρ_2, R denote the inclinations of the three lines to the given line. This theorem, which is equally applicable to Planimetry and Spherics, has been introduced to illustrate the use of the equation (A).

8. By this projection (§ 3) a plane n^{ic} is generally projected into a spherical $(2n)^{ic}$ with $n(n-1)$ nodes.

Let the quantic be arranged in order of its dimensions,

$$u_n + u_{n-1} + \ldots + u_0 = 0.$$

When transformed, it becomes in spherical coordinates

$$u_n + u_{n-1}(1+\sec\rho) + \ldots + u_0(1+\sec\rho)^n = 0.$$

This may be written $\qquad v_n + v_{n-1}\sec\rho = 0,$

or, when rationalised, $\qquad v_n^2 = v_{n-1}^2(1+x^2+y^2).$

The intersections of the curves v_n, v_{n-1} are nodes.

Thus the projection of a conic (not being a circle) is a binodal spherical quartic.

9. The projections of a circle and of circular quantics may have lower dimensions.

Let the plane circle be

$$x^2+y^2+2ax+2by+c = 0.$$

Its projection is the spherical circle

$$(c+1)\sec\rho + 2ax + 2by + c - 1 = 0.$$

As is known, the centres of the two circles are not in the line of projection.

Let a circular cubic, of which a focus is at the origin, be written

$$(x^2+y^2)\,u_1 + v_1 = 0.$$

Its projection is the spherical binodal quartic with two double cyclic arcs, one of which is the quadrantal polar of the origin,

$$(v_1-u_1)^2 = \sec^2\rho\,(v_1+u_1)^2.$$

10. In transforming quadrically from plane to spherical tangential coordinates, conical projection is not directly applicable.

In Fig. 1, let $P'F$ be the complement of $P'R$, or $p = \frac{\pi}{2} - \rho$.

Let also p' be the reciprocal of r with respect to the sphere.

Then $$\tan\left(\frac{\pi}{4}-\frac{p}{2}\right) = \tan\frac{\rho}{2} = \frac{r}{R} = \frac{R}{p'}.$$

The following formulæ of transformation are thus dual to those given in § 3,

$$\frac{R}{p'} = \tan\left(\frac{\pi}{4}-\frac{p}{2}\right) = \frac{\cot p}{\operatorname{cosec} p+1}, \quad \frac{R^2}{p'^2} = \frac{\operatorname{cosec} p-1}{\operatorname{cosec} p+1}.$$

Hence, if $R = 1$, the Boothian coordinates,

$$\xi\left(=\frac{1}{p'}\cos\theta\right), \quad \eta\left(=\frac{1}{p'}\sin\theta\right),$$

become in Spherics

$$\frac{\xi}{1+\operatorname{cosec} p}, \quad \frac{\eta}{1+\operatorname{cosec} p},$$

where

$$\operatorname{cosec}^2 p = 1+\xi^2+\eta^2.$$

To the reciprocal polar of the plane curve there corresponds, but not by conical projection, the quadrantal polar of the projected plane curve.

Thus, the reciprocal polar of a circle is a conic, whose focus is at the origin; the corresponding spherical curves are both small circles, complementary to each other.

11. A plane class-cubic, which has a double focus at the origin, is transformed, as in § 10, into a spherical class-quartic with two double foci, one of which is the origin, and two bitangents.

Usually, as in § 8, to a plane class-$(n)^{ic}$ there corresponds a spherical class-$(2n)^{ic}$ with $n(n-1)$ bitangents.

On Contangential and Coaxal Spherical Circles.

12. In Spherics, these systems of circles are dual or complementary to each other, *i.e.*, a point in one circle is the pole of a tangent arc to its dual circle, their radii being comemplents of each other. In Spherics, a circle cannot be considered geometrically or analytically, without its twin antipodal circle, *i.e.*, they must be treated together as the intersections of a sphere with a cone of twin-pair sheets, so that, whereas *in plano* there is but one real radical axis of two circles, the other being the line at infinity, in Spherics there are always two real radical axes of two circles, unless they are compolar or concentric. The radical axes of two small circles pass through the intersections of their compolar great circles. Hence it follows that the duals or quadrantal poles of the radical axes of two circles are the centres of similitude of the complementary circles, in which two pairs of common tangents intersect

each other. The four radical centres of triads of circles, constituted by three small circles and their antipodal circles, are the duals or poles of the four axes of similitude of the corresponding triads of complementary circles. See *infra*, §§ 26, 27.

The two limiting point-circles in a coaxal system are the duals of limiting great circles of the complementary contangential system ; in the first coaxal system the radical axis is external, in the second the centre of similitude is internal.

13. The portions of the tangents, which are common to two small circles, which are terminated at the points of contact, are bisected in their radical axes ; and the points of bisection are a quadrant apart.

13. The angle, under which two circles intersect each other, is bisected by the line joining either centre of similitude to a point of intersection ; and the two centres of similitude subtend a right angle at such a point, real in Spherics, but imaginary in Planimetry.

If two circles be referred to their common tangents, as axes of coordinates, their equations are, by § 4,

$$\pm \sec \rho = \cos c + (x+y) \sin c, \quad \pm \sec \rho = \cos d + (x+y) \sin d.$$

By combining them, we determine the two radical axes of the given circles and their antipodal circles,

$$\cos c \pm \cos d + (x+y)(\sin c \pm \sin d) = 0,$$

or, separately,
$$x+y-\tan \tfrac{1}{2}(c+d) = 0,$$
$$x+y+\cot \tfrac{1}{2}(c+d) = 0.$$

Remembering that x, y denote $\tan x$, $\tan y$, we establish the two parts of the theorem, when $x = 0$, $y = 0$ separately.

14. *In plano.* When the radius of the sphere is infinite,

$$x+y = \tfrac{1}{2}(c+d), \quad 1 = 0;$$

the theorem holds good for a single radical axis (Gergonne, tom. VIII., p. 323). For the dual theorem *in plano*, if the given circles intersect each other in two real points, as the forms of their equations imply, there is one, and but one, centre of similarity, and the second part of the theorem is inapplicable.

The next proposition is preliminary to the main property of radical axes.

15. If from any point P tangent arcs t_1, t_2 be drawn to two small circles, whose radii are δ_1, δ_2, and if perpendicular arcs p_1, p_2 be also drawn from P on their radical axes,

$$\cos^2 t_1 - \cos^2 t_2 = \Omega \sin p_1 \sin p_2,$$

$$\text{if } \Omega^2 = (\sec^2 \delta_1 - \sec^2 \delta_2)^2 + 4 \sec^2 \delta_1 \sec^2 \delta_1 \sin^2 O_1 O_2.$$

In plano, $t_1^2 - t_2^2 = 2p\,(O_1 O_2)$.

15. If a transversal intersect two small circles under angles t_1, t_2, and if perpendicular arcs p_1, p_2 be drawn from their centres of similitude on this great circle,

$$\cos^2 t_1 - \cos^2 t_2 = \Omega \sin p_1 \sin p_2,$$

$$\text{if } \Omega^2 = (\operatorname{cosec}^2 \delta_1 - \operatorname{cosec}^2 \delta_2)^2 + 4 \operatorname{cosec}^2 \delta_1 \operatorname{cosec}^2 \delta_2 \sin^2 O_1 O_2.$$

In plano,

$$\cos^2 t_1 - \cos^2 t_2 = p_1 p_2 \left(\frac{1}{\delta_1^2} - \frac{1}{\delta_2^2}\right).$$

If ρ be the radius vector of $P(x, y)$, (h_1, k_1), (h_2, k_2) denote the centres O_1, O_2, and T_1, T_2 the tangents drawn from the origin to the circles,

$$\cos^2 t_1 - \cos^2 t_2 = \sec^2 \delta_1 \cos^2 (O_1 P) - \sec^2 \delta_2 \cos^2 (O_2 P)$$
$$= \cos^2 \rho \left\{\cos^2 T_1 (1 + x h_1 + y k_1)^2 - \cos^2 T_2 (1 + x h_2 + y k_2)^2\right\}$$

(Gudermann's *Sphärik*, § 6; Graves' *Appendix to Chasles' Spherical Conics*, § 3)

$$= \cos^2 \rho \left\{\cos T_1 - \cos T_2 + x (h_1 \cos T_1 - h_2 \cos T_2) + y (k_1 \cos T_1 - k_2 \cos T_2)\right\}$$
$$\times \left\{\cos T_1 + \cos T_2 + x (h_1 \cos T_1 + h_2 \cos T_2) + y (k_1 \cos T_1 + k_2 \cos T_2)\right\}$$
$$= \Omega_1 \sin p_1 \sin p_2 \quad \text{(Gudermann, § 13; Graves, § 4).}$$

For $(\cos T_1 - \cos T_2)^2 + (h_1 \cos T_1 - h_2 \cos T_2)^2 + (k_1 \cos T_1 - k_2 \cos T_2)^2$

$$= \sec^2 r_1 \cos^2 T_1 + \sec^2 r_2 \cos^2 T_2 - 2 \sec r_1 \sec r_2 \cos T_1 \cos T_2 \cos (O_1 O_2)$$
$$= \sec^2 \delta_1 + \sec^2 \delta_2 - 2 \sec \delta_1 \sec \delta_2 \cos (O_1 O_2).$$

Cor. 1.—If the product $\sin p_1 \sin p_2$ be constant, *i.e.*, if P be a point in any sphero-conic, which has the radical axes of the given circles for its cyclic arcs, the difference of the squares of the cosines of the tangent arcs $(\cos^2 t_1 - \cos^2 t_2)$ is constant.

In plano the corresponding locus is a parallel to the radical axis.

Cor. 2.—Conversely, if the difference $(\cos^2 t_1 - \cos^2 t_2)$ has various constant values, the several loci of the point P are biconcyclic sphero-conics, whose cyclic arcs are the radical axes of the circles.

Cor. 3.—If X, Y, Z be three singly coaxal circles, the squares of the sines of the tangent arcs drawn from any point of one of them to the other two are in the ratio of multiples of the sines of the perpendiculars from that point on the separate radical axes;

$$\sin^2 t_2 \operatorname{cosec} p_2 : \sin^2 t_3 \operatorname{cosec} p_3 = \text{constant}.$$

16. The following is a Geometrical proof of the dual theorem.

Let P_1, P_2 be the perpendicular arcs from O_1, O_2 on the transversal; p_1, p_2 those drawn from Q_1, Q_2; δ_1, δ_2 the radii of the circles.

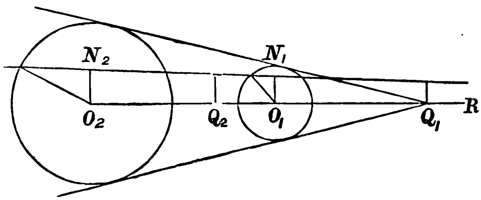


Fig. 3.

If Q_1R be eliminated from the ratios,

$$\sin P_1 \operatorname{cosec} O_1R = \sin P_2 \operatorname{cosec} O_2R = \sin p_1 \operatorname{cosec} Q_1R = \sin p_2 \operatorname{cosec} Q_2R,$$

$$\sin P_1 \sin Q_1Q_2 = \sin p_2 \sin O_1Q_1 + \sin p_1 \sin O_1Q_2,$$

$$\sin P_2 \sin Q_1Q_2 = \sin p_2 \sin O_2Q_1 - \sin p_1 \sin O_2Q_2.$$

Again, $$\sin \delta_1 \operatorname{cosec} O_1Q_1 = \sin \delta_2 \operatorname{cosec} O_2Q_1 = \Omega_1 \operatorname{cosec} O_1O_2,$$

$$\sin \delta_1 \operatorname{cosec} O_1Q_2 = \sin \delta_2 \operatorname{cosec} O_2Q_2 = \Omega_2 \operatorname{cosec} O_1O_2,$$

if $$\Omega_1^2 = \sin^2 \delta_1 + \sin^2 \delta_2 - 2 \sin \delta_1 \sin \delta_2 \cos O_1O_2,$$

$$\Omega_2^2 = \sin^2 \delta_1 + \sin^2 \delta_2 + 2 \sin \delta_1 \sin \delta_2 \cos O_1O_2.$$

Hence $$\Omega_1\Omega_2 \sin Q_1Q_2 = 2 \sin \delta_1 \sin \delta_2 \sin O_1O_2.$$

For

$$\frac{1}{\Omega_1} \sin \delta_1 = \operatorname{cosec} O_1O_2 \sin (O_2Q_1 - O_1O_2) = \frac{1}{\Omega_1} \sin \delta_2 \cos O_1O_2 - \cos O_2Q_1,$$

$$\frac{1}{\Omega_2} \sin \delta_1 = \operatorname{cosec} O_1O_2 \sin (O_1O_2 - O_2Q_2) = \cos O_2Q_2 - \frac{1}{\Omega_2} \sin \delta_2 \cos O_1O_2,$$

$$\Omega_1\Omega_2 \sin Q_1Q_2 \operatorname{cosec} O_1O_2$$

$$= \sin \delta_2 (\sin \delta_1 + \sin \delta_2 \cos O_1O_2) - \sin \delta_2 (\sin \delta_2 \cos O_1O_2 - \sin \delta_1)$$

$$= 2 \sin \delta_1 \sin \delta_2.$$

Consider the complements of the intersections t_1, t_2, and substitute the preceding values of $\sin P_1$, $\sin P_2$,

$$\cos^2 t_1 - \cos^2 t_2 = \sin^2 P_1 \operatorname{cosec}^2 \delta_1 - \sin^2 P_2 \operatorname{cosec}^2 \delta_2$$

$$= \frac{4}{\Omega_1 \Omega_2} \sin p_1 \sin p_2 \sin^2 O_1 O_2 \operatorname{cosec}^2 Q_1 Q_2$$

$$= \Omega_1 \Omega_2 \sin p_1 \sin p_2 \operatorname{cosec}^2 \delta_1 \operatorname{cosec}^2 \delta_2$$

$$= \sin p_1 \sin p_2 \sqrt{\{(\operatorname{cosec}^2 \delta_1 + \operatorname{cosec}^2 \delta_2)^2 - 4 \operatorname{cosec}^2 \delta_1 \operatorname{cosec}^2 \delta_2 \cos^2 O_1 O_2\}}.$$

Hence the plane theorem may be deduced, or it may be proved independently.

Cor. 1.—If the transversal be a tangent to a conic, spherical or plane, which has the centres of similitude for its foci, the difference of the squares of the cosines of the angles of intersection is constant.

Cor. 2.—If the transversal touch one of three singly contangential circles, the ratio $\frac{1}{p_2} \sin^2 t_2 : \frac{1}{p_3} \sin^2 t_3$ is constant.

There cannot be three bi-contangential or three bi-coaxal circles.

17. To determine the limiting circles of a coaxal system.

They are definite or point-circles for the lowest limit, and the radical axis in all cases for the superior limit.

By taking the radical axis as the (y) axis of Gudermann's rectangular coordinates, any of the coaxal circles may be thus denoted in one or other of the two coaxal systems:

$$(1)\ \sec \rho \cos k = 1 + hx, \text{ or } (2)\ \sec \rho = \cos k\,(1 + hx),$$

according as the circles do or do not meet their radical axis.

In these equations, h is the tangent of the abscissa of the centre; and k in (1) either ordinate of two fixed points, or in (2) the tangent from the origin.

If x, y denote, as usual, $\tan x$, $\tan y$,

$$\sec^2 \rho = 1 + x^2 + y^2.$$

In the system (1), the limits are a small circle whose centre is the origin, and angular radius k, and the radical axis.

In the system (2), point-circles ($\pm k$, 0) on the abscissa form the lowest limit.

In this case $h = \pm k$, $\sec \rho = \cos k\,(1 \pm kx)$,

or, since k denotes $\tan k$,

$$y^2 + (\sin k \pm x \cos k)^2 = 0.$$

The radical axis is the highest limit of the system.

18. To determine the limiting circles of a contangential system of circles.

They are the centre of similitude and for the highest limit definite, small, or great circles.

The equations

$$(1)\ \operatorname{cosec} p \sin \omega = 1 - h\xi, \quad (2)\ \operatorname{cosec} p = \cos t\,(1 - h\xi),$$

which are the duals of those given in § 17, denote two contangential systems; in (1) the centre of similitude is external, in (2) internal, and the common tangents are imaginary.

In dualising, $x + \xi' = 0$, $y + \eta' = 0$, where ξ', η' denote the cotangents of the intercepts on the coordinate arcs of a line (ξ', η'), which is the quadrantal polar of the point (x, y),

$$\operatorname{cosec}^2 p = 1 + \xi^2 + \eta^2;$$

of the constants, h ($\equiv \tan h$) is unaltered with the centre.

In (1), 2ω = mutual inclination of a pair of common tangents, which meet in the exterior centre of similitude.

In (2), $t = OPT = OP'T'$.

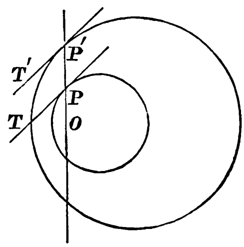


FIG. 4.

In Fig. 4, the circles are drawn *in plano*, which have O for an internal centre of similitude; TP is parallel to $T'P'$, and OPP' is perpendicular to the line of centres.

In Spherics, OPP' is the quadrantal polar of the origin, and O, the internal centre of similitude, is 90° distant from the origin.

The superior limits of the system are, in (1), $p = \omega$, the small circle, whose centre is the origin and diameter (2ω), and in (2), great circles,

$$\left\{\eta^2 + (\cos t \pm \xi \sin t)^2 = 0\right\},$$

which are (t) distant from the centre of similitude on either side.

In Plane Geometry, the limiting contangential circles are infinite, as also appears from considering Fig. 4, since there is no finite limit to the system.

The inferior limits are the centres of similitude ($\xi = 0$).

19. Any circle which passes through the limiting point-circles of the coaxal system (2), and has its centre in their radical axis, is an orthogonal trajectory of that system.

19. Any small circle which touches the limiting great circles of the contangential system (2), and whose centre is equidistant from those great circles, has common tangents with circles of the system (2), quadrants in length.

A circle which passes through the limiting point-circles of a coaxal system (2), and whose centre is in the (y) axis of coordinates, belongs to the coaxal system (1),

$$\sec\rho\cos k = 1 + ly.$$

It cuts orthogonally any circle of the system (2),

$$\sec\rho = \cos k\,(1+hx).$$

For the condition of coorthotomy is satisfied:

$$\sec\delta_1 \sec\delta_2 = \sec k\sqrt{(1+l^2)}\cos k\sqrt{(1+h^2)} = \sec(O_1O_2),$$

if δ_1, δ_2 are the radii, O_1, O_2 the centres of the two circles.

The following theorem is required for the next section.

20. If from any point a transversal be drawn to meet a fixed spherical circle,

$$\tan\frac{\rho_1}{2}\tan\frac{\rho_2}{2} = \tan^2\frac{t}{2},$$

where ρ_1, ρ_2 denote the intercepts from that point, and t the tangent arc, if the point is external. Hymers' *Spherical Trig.*, p. 37.

20. If from any point in a line intersecting a fixed circle, plane or spherical, two tangents be drawn,

$$\tan\frac{\rho_1}{2}\tan\frac{\rho_2}{2} = \tan^2\frac{t}{2},$$

where ρ_1, ρ_2 denote their inclinations to the initial line, and t the inclination of a tangent at either intersection to that line.

This theorem may be deduced at once by conical projection from the plane property $$r_1 r_2 = t^2,$$

or it may be derived from the equation (§ 4)

$$\sec\rho = \cos t + (x+y)\sin t.$$

In the dual theorem, care must be taken to measure the angles ρ_1, ρ_2 in the same direction.

On Similitude and Inversion.

21. These relations are closely allied, since a centre of similitude is also a centre of inversion, for two circles.

If through a centre of similitude a vector arc be drawn to intersect two circles at

If from each point of a radical axis of two circles, plane or spherical, two tangents be drawn

distances $\rho_1, \rho_2, \rho_3, \rho_4,$

$$\tan\frac{\rho_1}{2} : \tan\frac{\rho_3}{2} :: \tan\frac{\rho_2}{2} : \tan\frac{\rho_4}{2} :: \tan\frac{c}{2} : \tan\frac{d}{2}.$$

to them, the ratio of the tangents of half the angles, which they make with the radical axis, is constant. (Chasles' *Géométrie Supérieure*, p. 468.)

(The tangent arcs are c, d.)

This theorem defines spherical similitude, and may be deduced by conical projection (§ 3) from the plane analogue.

By dualising, a second species of similitude is brought to light, in which the radical axis is also an axis of dual inversion.

A formal proof is obtained by the aid of § 4.

22. Let the two circles be referred to their centre of similitude, not necessarily being external to both,

$$(1)\ \sec\rho = \cos c + (x+y)\sin c, \qquad (2)\ \sec\rho = \cos d + (x+y)\sin d,$$

if c, d denote the intercepts on a common tangent arc.

For any transversal through the origin, both in Planimetry and Spherics, as is seen by gnomonic projection,

$$x \operatorname{cosec}\beta = y \operatorname{cosec}\alpha = \rho \operatorname{cosec} 2\omega,$$

if x, y, ρ in Spherics denote tangents, and $\alpha+\beta = 2\omega$.

Where it intersects (1),

$$\sec^2\rho = 1 + x^2 \sin^2 2\omega \operatorname{cosec}^2\beta$$

$$= \left\{\cos c + x(\sin\alpha + \sin\beta)\sin c \operatorname{cosec}\beta\right\}^2.$$

Hence $$\left\{\sin\beta\sin c - x(\sin\alpha+\sin\beta)\cos c\right\}^2$$

$$= 4x^2\sin^2\omega\left\{\cos^2\tfrac{1}{2}(\alpha-\beta) - \cos^2\tfrac{1}{2}(\alpha+\beta)\right\}$$

$$= 4x^2 \sin\alpha\sin\beta\sin^2\omega.$$

And $$2x\sin\omega\left\{\cos c\cos\tfrac{1}{2}(\alpha-\beta) \pm \sqrt{(\sin\alpha\sin\beta)}\right\} = \sin c\sin\beta,$$

$$\sec\rho\left\{\cos c\cos\tfrac{1}{2}(\alpha-\beta) \pm \sqrt{(\sin\alpha\sin\beta)}\right\}$$

$$= \cos\tfrac{1}{2}(\alpha-\beta) \pm \cos c\sqrt{(\sin\alpha\sin\beta)}.$$

If ρ_1, ρ_2 denote the vector arcs intercepted by the first circle,

$$\tan\frac{\rho_1}{2} = \frac{\tan\rho_1}{1+\sec\rho_1} = \tan\frac{c}{2}\cos\omega\left\{\cos\tfrac{1}{2}(\alpha-\beta)+\surd(\sin\alpha\sin\beta)\right\}$$

$$= \tan\frac{c}{2}\sec\omega\left\{\sqrt{\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right)}-\sqrt{\left(\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)}\right\}^2,$$

$$\tan\frac{\rho_2}{2} = \tan\frac{c}{2}\sec\omega\left\{\sqrt{\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\right)}+\sqrt{\left(\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)}\right\}^2.$$

If ρ_3, ρ_4 denote the vector arcs for the second circle, they have similar values, so that

$$\tan\frac{\rho_1}{2} : \tan\frac{\rho_3}{2} :: \tan\frac{\rho_2}{2} : \tan\frac{\rho_4}{2} :: \tan\frac{c}{2} : \tan\frac{d}{2}.$$

Cor. 1. $\tan\frac{\rho_1}{2}\tan\frac{\rho_2}{2} = \tan^2\frac{c}{2}$, as in § 20.

Cor. 2. $$\frac{\sqrt{\left(\tan\frac{\rho_2}{2}\right)}+\sqrt{\left(\tan\frac{\rho_1}{2}\right)}}{\sqrt{\left(\tan\frac{\rho_2}{2}\right)}-\sqrt{\left(\tan\frac{\rho_1}{2}\right)}} = \sqrt{\left(\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\right)}.$$

Cor. 3. If $\tan^2\frac{R}{2} = \tan\frac{c}{2}\tan\frac{d}{2}$, the circle, whose equation is

$$\sec R = \cos\tfrac{1}{2}(c-d)\sec\tfrac{1}{2}(c+d),$$

is coaxal with the two given circles, and has the common radical axis

$$x+y = \tan\tfrac{1}{2}(c+d).$$

This is also obvious geometrically, when the given circles intersect each other; for their circle of inversion must then pass through their intersections.

Cor. 4. Since

$$\tan\frac{\rho_1}{2}\tan\frac{\rho_4}{2} = \tan\frac{\rho_2}{2}\tan\frac{\rho_3}{2} = \tan^2\frac{R}{2},$$

the centre of similitude is the centre of a system of points in involution, whose foci are on the circle of inversion.

23. *On Inversion.* — A centre of similitude of two contangential circles is also their centre

23. *On Dual Inversion.* — A radical axis of two circles is also an axis of dual inversion of those

of inversion, and the circle (R) of § 22 is their circle of inversion.

circles, and is such that, if from any point therein two pairs of tangents be drawn, making angles ρ_1, ρ_2; ρ_3, ρ_4 with the axis, then in cross order

$$\tan\frac{\rho_1}{2}\tan\frac{\rho_4}{2} = \tan\frac{\rho_2}{2}\tan\frac{\rho_3}{2} = \tan^2\frac{R}{2}.$$

The circle (R) is compolar with the radical axis. See § 41.

24. An angle is unaltered by inversion.

24. In Spherics, an arc of a great circle, and in Planimetry a straight line, is unaltered in length by Dual Inversion. See § 41.

In Planimetry an angle is unaltered by inversion; by conical projection, the same theorem is established in Spherics.

A formal proof is subjoined.

Let a pair of circles be written

$$(1)\ \ c\sec r = 1+mx+ny, \quad (2)\ \ c'\sec r = 1+m'x+n'y.$$

The tangents at their points of intersection are

$$c\,(1+xx'+yy') = (1+mx+ny)\sec r',$$
$$c'\,(1+xx'+yy') = (1+m'x+n'y)\sec r'.$$

If θ denote their mutual inclination, by Gudermann's *Sphärik*, § 11,

$$\cos\theta = \frac{P}{QR},$$

where
$$\begin{aligned} P &= (cx'-m\sec r')(c'x'-m'\sec r')+(cy'-n\sec r')(c'y'-n'\sec r') \\ &\quad +(c-\sec r')(c'-\sec r') \\ &= cc'\,(1+x'^2+y'^2)+(mm'+nn'+1)\sec^2 r' \\ &\quad -\left[(mc'+m'c)\,x'+(nc'+n'c)\,y'+c+c'\right]\sec r' \\ &= (mm'+nn'-cc'+1)\sec^2 r', \\ Q^2 &= (cx'-m\sec r')^2+(cy'-n\sec r')^2+(c-\sec r')^2 \\ &= (m^2+n^2-c^2+1)\sec^2 r', \\ R^2 &= (m'^2+n'^2-c'^2+1)\sec^2 r'. \end{aligned}$$

Hence

$$(mm'+nn'-cc'+1)\sec\theta = (m^2+n^2-c^2+1)^{\frac{1}{2}}\,(m'^2+n'^2-c'^2+1)^{\frac{1}{2}}.$$

Next, to obtain the inverse circles.

Write (1) in the form

$$c\left(1+\tan^2\frac{r}{2}\right) = 1-\tan^2\frac{r}{2}+2\,(m\cos\theta+n\sin\theta)\tan\frac{r}{2}.$$

Its inverse is, if $\tan\dfrac{r}{2}\tan\dfrac{\rho}{2}=\kappa,$

$$\sec\rho\left\{\kappa^2(c+1)+c-1\right\} = c-1-\kappa^2(c+1)+2\kappa\,(mx+ny).$$

The inverse of (2) is

$$\sec\rho\left\{\kappa^2(c'+1)+c'-1\right\} = c'-1-\kappa^2(c'+1)+2\kappa\,(m'x+n'y).$$

The function P becomes for these values

$$4\kappa^2(mm'+nn')+\left\{(c-1)-\kappa^2(c+1)\right\}\left\{(c'-1)-\kappa^2(c'+1)\right\}$$
$$-\left\{(c-1)+\kappa^2(c+1)\right\}\left\{(c'-1)+\kappa^2(c'+1)\right\},$$

or $\quad 4\kappa^2(mm'+nn'-cc'+1).$

Similarly, $\quad Q^2$ becomes $4\kappa^2(m^2+n^2-c^2+1),$

$\quad R^2 \quad$,, $\quad 4\kappa^2(m'^2+n'^2-c'^2+1).$

The angle of intersection (θ) is therefore unaltered by inversion.

25. If through a centre of similitude any two arcs be drawn meeting the first circle in the points R, R', S, S', and the second in the points ρ, ρ', σ, σ', then the arcs RS, $\rho\sigma$; $R'S'$, $\rho'\sigma'$ will meet on one radical axis; and the arcs RS, $\rho'\sigma'$; $R'S'$, $\rho\sigma$ will meet on the other radical axis of the two circles.

25. If from any two points in a radical axis two pairs of tangent arcs be drawn, so as to constitute two quadrilaterals with a common diagonal arc,—the arcs which join the angular points, one taken from each quadrilateral, will pass through one or other centre of similitude.

The two quadrilaterals $RR'SS'$, $\rho\rho'\sigma\sigma'$ have O for a common centre, and the other two pairs of centres are collinear with O, the lines of collinearity forming a harmonic pencil with the other collinear sides $OR\rho$, $OS\sigma$. See Fig. in Salmon's *Conic Sections*, p. 107.

As in § 22, let the two circles be referred to the common tangents

(1) $\sec\rho = \cos c + (x+y)\sin c$, (2) $\sec\rho = \cos d + (x+y)\sin d$.

For any transversal $O\rho R$,

$$x \operatorname{cosec}\beta = y \operatorname{cosec}\alpha = \rho \operatorname{cosec} 2\omega,$$

where it intersects (1), as in § 22,

$$\frac{1}{\rho}\sin c\cos\omega = \cos c\cos\tfrac{1}{2}(\alpha-\beta) \pm \sqrt{(\sin\alpha\sin\beta)}\ ;$$

the two values of ρ denote $\tan O\rho'$, $\tan O\rho$.

So also are denoted $\tan OR'$, $\tan OR$, when d is written for c.

For a second transversal $O\sigma S$,

$$x \operatorname{cosec}\delta = y \operatorname{cosec}\gamma = \rho \operatorname{cosec} 2\omega\ ;$$

$$\frac{1}{\rho'}\sin c\cos\omega = \cos c\cos\tfrac{1}{2}(\gamma-\delta) \pm \sqrt{(\sin\gamma\sin\delta)},$$

whose two values denote $\tan O\sigma'$, $\tan O\sigma$. Similarly, we express $\tan OS'$, $\tan OS$.

The equation to the chord $\rho'\sigma'$ is

$$(x-\rho_1\sin\beta\operatorname{cosec}2\omega)(\rho_1\sin\alpha-\rho_1'\sin\gamma)$$
$$= (y-\rho_1\sin\alpha\operatorname{cosec}2\omega)(\rho_1\sin\beta-\rho_1'\sin\delta),$$

or $$x\left(\frac{1}{\rho_1'}\sin\alpha-\frac{1}{\rho_1}\sin\gamma\right)-y\left(\frac{1}{\rho_1'}\sin\beta-\frac{1}{\rho_1}\sin\delta\right)$$
$$= (\sin\alpha\sin\delta-\sin\beta\sin\gamma)\operatorname{cosec}2\omega \ldots\ldots(3).$$

The equation to the chord RS is

$$x\left(\frac{1}{R_2'}\sin\alpha-\frac{1}{R_2}\sin\gamma\right)-y\left(\frac{1}{R_2'}\sin\beta-\frac{1}{R_2}\sin\delta\right)$$
$$= (\sin\alpha\sin\delta-\sin\beta\sin\gamma)\operatorname{cosec}2\omega \ldots\ldots(4).$$

Multiply (3) by $\sin c$, (4) by $\sin d$, and add:

$$\sec\omega(\cos c+\cos d)\left\{x\left[\sin\alpha\cos\tfrac{1}{2}(\gamma-\delta)-\sin\gamma\cos\tfrac{1}{2}(\alpha-\beta)\right]\right.$$
$$\left.-y\left[\sin\beta\cos\tfrac{1}{2}(\gamma-\delta)-\sin\delta\cos\tfrac{1}{2}(\alpha-\beta)\right]\right\}$$
$$= \operatorname{cosec}2\omega(\sin c+\sin d)(\sin\alpha\sin\delta-\sin\beta\sin\gamma)\ \ldots\ldots\ldots\ldots(5).$$

But, since $\alpha+\beta=\gamma+\delta=2\omega$, the coefficient of x is

$$\sin\alpha\cos(\gamma-\omega)-\sin\gamma\cos(\alpha-\omega)=\sin(\alpha-\gamma)\cos\omega,$$

and $$\sin\alpha\sin(2\omega-\gamma)-\sin\gamma\sin(2\omega-\alpha)=\sin 2\omega\sin(\alpha-\gamma).$$

After these and like reductions, (5) takes the form of the radical axis, in which (3) and (4) meet (see § 13),

$$x+y=\tan\tfrac{1}{2}(c+d) \quad\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Similarly, it may be shown that $\rho\sigma$ and $R'S'$ intersect in (5).

The equation to $R'S'$ is

$$x\left(\frac{1}{R_1'}\sin\alpha-\frac{1}{R_1}\sin\gamma\right)-y\left(\frac{1}{R_1'}\sin\beta-\frac{1}{R_1}\sin\delta\right)$$
$$=(\sin\alpha\sin\delta-\sin\beta\sin\gamma)\operatorname{cosec}2\omega \ldots\ldots(6).$$

Multiply (3) by $\sin c$, (6) by $\sin\delta$, and subtract one from the other.

The resulting equation may be reduced, as (5) was, to denote the second radical axis (see § 13),

$$x+y+\cot\tfrac{1}{2}(c+d)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Similarly, it may be shown that $\rho'\sigma'$ and RS meet in (7).

The centres of the two quadrilaterals lie in the two lines

$$x\sqrt{(\sin\alpha\sin\gamma)}\pm y\sqrt{(\sin\beta\sin\delta)}=0.$$

Cor.—The tangents at R, ρ; R', ρ' meet on one radical axis; those at R, ρ'; R', ρ meet on the other radical axis.

Cor.—Arcs which join the points of contact of a pair of tangents, which are drawn from any point in the radical axis to one circle, to the points of contact of a pair of tangents to the second circle, intersect in one or other centre of similitude.

On the Problem of Contacts.

26. To describe pairs of spherical circles, which shall touch three given circles and their antipodal circles, by triads.

Let the radical centre of the three given circles be taken as the origin of Gudermann's coordinates; their equations may be written

$$\left.\begin{aligned}\sec t\sec\rho&=1+ax+by\\ \sec t\sec\rho&=1+cx+dy\\ \sec t\sec\rho&=1+ex+fy\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The centres or poles are (a, b), (c, d), (e, f), severally; t is the tangent arc drawn from the origin to any of the three circles, supposed to be external to each other (see Fig. in Salmon's *Conics*, p. 112).

Also $\sec^2\rho = 1+x^2+y^2$, if $x, y, a, b, \ldots$ denote the tangents of those arcs, as usual. The Jacobian of the three circles,

$$\sec\rho = \sec t,$$

cuts them orthogonally, since it satisfies the test of coorthotomy (*Proceedings*, Vol. XVI., p. 111).

The centre of the Jacobian is also the centre of similitude of a required pair of circumscribed circles (Salmon, *loc. cit.*), and therefore the Jacobian is their circle of inversion, and therefore also coaxal with them, their common radical axis being the axis of similitude of the three given circles. (Casey, *Sequel to Euclid*, p. 118.)

There are four radical centres, including the origin,

$$ax+by+1 = \pm(cx+dy+1) = \pm(ex+fy+1);$$

but at present only the positive signs are considered. And there are four corresponding Jacobian orthotomic circles, of which that which cuts the three given circles is

$$\sec\rho = \sec t.$$

The equivalent Boothian tangential equations to the given circles have the form

$$\sin^2 r_1 (a^2+b^2+1)(\xi^2+\eta^2+1) = (a\xi+b\eta-1)^2,$$

or, since $$a^2+b^2+1 = \sec^2 t \sec^2 r_1,$$

$$\tan^2 r_1 \sec^2 t\,(\xi^2+\eta^2+1) = (a\xi+b\eta-1)^2.$$

There are therefore four axes of similitude, thus denoted

$$\pm\theta \sec t \tan r_1 + a\xi'+b\eta'-1 = 0,$$

$$\pm\theta \sec t \tan r_2 + c\xi'+d\eta'-1 = 0,$$

$$\pm\theta \sec t \tan r_3 + e\xi'+f\eta'-1 = 0.$$

The set of values here required is found by giving the same sign to $\tan r_1$, $\tan r_2$, $\tan r_3$,

$$\frac{\theta\sec t}{|a,\ d,\ 1|} = \frac{-\xi'}{|\tan r_1,\ d,\ 1|} = \frac{\eta'}{|\tan r_1,\ c,\ 1|} = \frac{1}{|\tan r_1,\ c,\ f|}.$$

By the preceding theorems, the required equation to a pair of cir-

cumscribed circles must have the form

$$\sec\rho = \sec t + \mu\,(x\xi' + y\eta' - 1).$$

But it is better to introduce first another circle coorthotomic with the Jacobian, and having the axis of similitude (ξ', η') for their radical axis,

$$\sec\rho = \cos t + \sin t \tan t\,(x\xi' + y\eta').$$

We will now thus present the required pair of circles,

$$(1+\tau)\sec\rho = \cos t + \tau \sec t + \sin t \tan t\,(x\xi' + y\eta') \quad \text{......(2).}$$

The multiplier (τ) must be found from the condition that the pair touch each of the three given circles.

In general, for the condition of contact for two circles, whose spherical radii are r_1, R, and whose equations are

$$\cos r_1 \sec\rho\,(1+a^2+b^2)^{\frac{1}{2}} = 1 + ax + by,$$

$$\cos R \sec\rho\,(1+\alpha^2+\beta^2)^{\frac{1}{2}} = 1 + \alpha x + \beta y,$$

$$\cos(r_1 \sim R) = \cos D = (1 + a\alpha + b\beta)(1+a^2+b^2)^{-\frac{1}{2}}(1+\alpha^2+\beta^2)^{-\frac{1}{2}}.$$

Let these conditions be applied to (2) and the first circle of (1).

In this case $\qquad a^2+b^2+1 = \sec^2 t \sec^2 r_1,$

$$(1+\alpha^2+\beta^2)^{\frac{1}{2}}(\cos t + \tau \sec t) = (1+\tau)\sec R \quad \text{..........(3).}$$

With these values, the condition of contact becomes

$$(1+\tau)\tan r_1 \tan R = (a\xi' + b\eta' - 1)\sin^2 t$$

$$= -\theta \sin^2 t \sec t \tan r_1 \quad \text{..........(4).}$$

Eliminate R from (3) and (4),

$$(1+\tau)^2 + \theta^2 \sin^2 t \tan^2 t = (\cos t + \tau \sec t)^2 + (\xi'^2 + \eta'^2)\sin^2 t \tan^2 t.$$

Hence, finally, the two values of τ in (2) are obtained in terms of known constants, $\quad \tau^2 = \cos^2 t - (\xi'^2 + \eta'^2 - \theta^2)\sin^2 t.$

This equation (2) and the values of τ have been otherwise obtained by Gergonne's process (Salmon's *Conics*, p. 110).

But since different radical centres (other than the origin) and Jacobians are necessary for the other cases, where one or more of the antipodal circles are concerned, three-point coordinates are preferable to exhibit the equations collectively.

27. To find the equations to pairs of spherical circles, which shall

touch three given small circles, and their antipodal circles, when three point-coordinates are employed.

Let ABC, the triangle of reference, have the centres of the given circles for its angular points; if a, b, c; α, β, γ denote the sines of the sides and the perpendicular coordinates, and if r_1, r_2, r_3 be the angular radii, and $6V = bc \sin A$ (in Todhunter's *Spherical Trig.*, p. 22, $n = 3V$). For the given circles and their antipodal circles,

$$\left.\begin{aligned} S_1 &\equiv a\alpha + b\beta\cos c + c\gamma\cos b \mp 6V\cos r_1 = 0 \\ S_2 &\equiv a\alpha\cos c + b\beta + c\gamma\cos a \mp 6V\cos r_2 = 0 \\ S_3 &\equiv a\alpha\cos b + b\beta\cos a + c\gamma \mp 6V\cos r_3 = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(1).$$

The equations are rendered homogeneous by the fundamental relation (see Salmon's *Solid Geometry*, *Sphero-Conics*, p. 198, where three-point and three-line coordinates are introduced),

$$\Sigma\,(a^2\alpha^2 + 2bc\,\beta\gamma\cos a) = (6V)^2.$$

The given circles have four radical centres, which are also the centres of the corresponding Jacobians,

$$\begin{aligned}(a\alpha + b\beta\cos c + c\gamma\cos b)\sec r_1 &= \pm(a\alpha\cos c + b\beta + c\gamma\cos a)\sec r_2 \\ &= \pm(a\alpha\cos b + b\beta\cos a + c\gamma)\sec r_3.\end{aligned}$$

The Jacobians of the several triads of circles are

$$\pm\, a\alpha\cos r_1 \pm b\beta\cos r_2 \pm c\gamma\cos r_3 = 6V,$$

or
$$a\alpha S_1 + b\beta S_2 + c\gamma S_3 = 0.$$

The Jacobians intersect the several triads orthogonally, for it was shown (*Proceedings*, Vol. XVI., p. 111) that the condition of coorthotomy for two circles, $\left[l\alpha + m\beta + n\gamma = d,\ l'\alpha + m'\beta + n'\gamma = d'\right]$,

is
$$\Sigma\,\{ll' - (m'n + mn')\cos A\} = dd'.$$

These conditions are satisfied by the Jacobians.

The tangential equivalents to the circles (1) are

$$p^2 = r_1^2; \quad q^2 = r_2^2; \quad r^2 = r_3^2;$$

if p, q, r denote the sines of the perpendiculars drawn from A, B, C on a tangent arc. For the four axes of similitude,

$$\frac{p^2}{r_1^2} = \frac{q^2}{r_2^2} = \frac{r^2}{r_3^2};$$

or, in point-coordinates, $a\alpha r_1 \pm b\beta r_2 \pm c\gamma r_3 = 0$,

also r_1, r_2, r_3 now denote $\sin r_1, \ldots$.

As has been stated in § 26, the required equation to a pair of circumscribed circles is coaxal with a Jacobian, the radical axis being the axis of similitude of the corresponding triad of circles,

$$a\alpha(\cos r_1+\tau\sin r_1)+b\beta(\cos r_2+\tau\sin r_2)+c\gamma(\cos r_3+\tau\sin r_3)=6V\ldots(3).$$

The two values of the multiplier (τ) are found by the condition that this circle (3) touches any one of the circles (1).

In general, if ρ_1, ρ_2 be the angular radii of two touching circles, and D be the connector of their centres,

$$\cos(\rho_1\pm\rho_2)=\cos D,$$

or
$$(\cos\rho_1\cos\rho_2-\cos D)^2=(1-\cos^2\rho_1)(1-\cos^2\rho_2).$$

If, then, their equations be

$$l\alpha+m\beta+n\gamma=d,\quad l'\alpha+m'\beta+n'\gamma=d',$$

the necessary condition for contact is

$$\left\{\Sigma\left[ll'-(mn'+m'n)\cos A\right]-dd'\right\}^2$$
$$=\left\{\Sigma(l^2-2mn\cos A)-d^2\right\}\left\{\Sigma(l'^2-2m'n'\cos A)-d'^2\right\}.$$

If this condition be applied in this case,

$$(1+\tau^2)(6V)^2$$
$$=\Sigma\left\{a^2(\cos r_1+\tau\sin r_1)^2-2bc(\cos r_2+\tau\sin r_2)(\cos r_3+\tau\sin r_3)\cos A\right\}.$$

Since there are four radical centres, and four axes of similitude, there are in all 16 pairs of circumscribed circles, or 32 solutions in all.

Cor. 1. If $r_1=r_2=r_3=0$, there is no axis of similitude, and the Jacobians denote the four circles described about ABC and its associated triangles, $A'BC$, $AB'C$, ABC', whose equations are

$$\pm a\alpha\pm b\beta\pm c\gamma=6V.$$

Cor. 2. If $r_1=r_2=r_3=\frac{1}{2}\pi$, the Jacobian is the imaginary circle ($6V=0$), $\tau^2=\cot^2 R$, if R be the radius of circumscribed circle; and the equation is derived to the circle inscribed in the polar triangle of ABC,

$$a\alpha+b\beta+c\gamma=6V\tan R,$$

whose angular radius is therefore the complement of R. (Todhunter's *Spher. Trig.*, p. 66.)

28. Dr. Casey has pointed out that each pair of circles circumscribed about a triad of circles is a degenerate case of spherocyclides, when the dirigent conic has double contact with the Jacobian.

In the *Proceedings of the Royal Irish Academy*, Vol. IX., p. 396, he obtained the equation to such a pair of circles,

$$\sin\frac{l}{2}\surd(S_1\sec r_1)+\sin\frac{m}{2}\surd(S_2\sec r_2)+\sin\frac{n}{2}\surd(S_3\sec r_3)=0,$$

where l, m, n denote the lengths of tangent arcs common to pairs of the given triad of circles, and S_1, S_2, S_3 denote their equations given in § 27.

I propose to obtain Dr. Casey's equation both analytically and geometrically.

29. In general, if the variable circle, which generates a sphero-cyclide, be

$$a\alpha L+b\beta M+c\gamma N=6V \quad\text{.........................(1)},$$

and the Jacobian of three given circles be, as in § 27,

$$a\alpha\cos r_1+b\beta\cos r_2+c\gamma\cos r_3=6V \quad\text{................(2)},$$

then, if the pole of (1), which moves so as to intersect (2) orthogonally, be made to lie on the dirigent conic

$$(u,\ v,\ w,\ u',\ v',\ w' \between p,\ q,\ r)^2=0,$$

the sphero-cyclide will be denoted by the equation

$$(u,\ v,\ w,\ u',\ v',\ w' \between S_1,\ S_2,\ S_3)^2=0.$$

For, by a known theorem of Dr. Casey, quadric transformation is thus effected from the tangential coordinates

$$p:q:r::S_1:S_2:S_3.$$

(Casey on *Cyclides*, § 40. See also a proof in *Proceedings*, Vol. XVI., p. 115).

30. To determine the dirigent conic, when it has double contact with the Jacobian.

Let the given triad of circles (1), in § 27, be supposed to intersect each other at the angles ϕ_1, ϕ_2, ϕ_3; then, if $(\widehat{S_2, S_3})=\phi_1$,

$$\cos a=\cos r_2\cos r_3+\sin r_2\sin r_3\cos(\pi-\phi_1)$$

$$=\cos(r_2-r_3)-2\sin r_2\sin r_3\cos^2\frac{\phi_1}{2}.$$

Next, let l, m, n denote, as in § 28, common tangent arcs,

$$\cos a=\sin r_2\sin r_3+\cos r_2\cos r_3\cos l$$

$$=\cos(r_2-r_3)-2\cos r_2\cos r_3\sin^2\frac{l}{2}.$$

Now, remembering that $(6V)^2 = \Sigma\,(a^2\alpha^2 + 2bc\beta\gamma\cos a)$,

the Jacobian may be written

$$a^2\alpha^2\sin^2 r_1 + b^2\beta^2\sin^2 r_2 + \ldots - 2bc\beta\gamma\sin r_2\sin r_3\cos\phi_1 - \ldots = 0,$$

or $$(a\alpha\sin r_1 + b\beta\sin r_2 + c\gamma\sin r_3)^2 = 4bc\beta\gamma\sin r_2\sin r_3\cos^2\frac{\phi_1}{2} + \ldots\ldots.$$

Since the dirigent conic has double contact with the Jacobian, its form must be

$$\frac{1}{a\alpha}\operatorname{cosec} r_1\cos^2\frac{\phi_1}{2} + \frac{1}{b\beta}\operatorname{cosec} r_2\cos^2\frac{\phi}{2} + \frac{1}{c\gamma}\operatorname{cosec} r_3\cos^2\frac{\phi_3}{2} = 0,$$

or, since $$\cos r_2\cos r_3\sin^2\frac{l}{2} = \sin r_2\sin r_3\cos^2\frac{\phi_1}{2},$$

$$\frac{1}{a\alpha}\sec r_1\sin^2\frac{l}{2} + \frac{1}{b\beta}\sec r_2\sin^2\frac{m}{2} + \frac{1}{c\gamma}\sec r_3\sin^2\frac{n}{2} = 0.$$

Its equivalent tangential equation is

$$\sin\frac{l}{2}\sqrt{(p\sec r_1)} + \sin\frac{m}{2}\sqrt{(p\sec r_2)} + \sin\frac{n}{2}\sqrt{(r\sec r_3)} = 0.$$

Hence Dr. Casey's equation to a pair of circumscribed circles is derived by the theorem quoted in § 29:

$$\sin\frac{l}{2}\sqrt{(S_1\sec r_1)} + \sin\frac{m}{2}\sqrt{(S_2\sec r_2)} + \sin\frac{n}{2}\sqrt{(S_3\sec r_3)} = 0.$$

Before I proceed to obtain this equation geometrically, the following lemmas are prefixed.

31. If two spherical circles be inverted from an arbitrary point, the product $\sin\frac{AB}{2}(\cot r_1\cot r_2)^{\frac{1}{2}}$ is unaltered, if AB denote a common tangent arc. This expresses in another form that the angle, under which two small circles intersect, is unaltered by inversion. (§ 24).

From § 30, $$\cos^2\frac{\phi_3}{2} = \cot r_1\cot r_2\sin^2\frac{c}{2}.$$

32. If A, B, C, D be four points on a spherical great or small circle,

$$\sin\frac{BC}{2}\sin\frac{AD}{2} + \sin\frac{CA}{2}\sin\frac{BD}{2} + \sin\frac{AB}{2}\sin\frac{CD}{2} = 0.$$

This extension of Ptolemy's theorem may be readily established, by joining the points to the centre, and expressing the semi-arcs in

terms of the angles, which they subtend. Identically

$$\sin\frac{a}{2}\sin\frac{b-c}{2}+\sin\frac{b}{2}\sin\frac{c-a}{2}+\sin\frac{c}{2}\sin\frac{a-b}{2}=0.$$

Let four circles whose radii are r_1, r_2, r_3, r_4 touch the preceding great circle $ABCD$ in the points A, B, C, D; then, if the system be inverted from any point, by § 31,

$$\sin\frac{B'C'}{2}\sin\frac{A'D'}{2}+\sin\frac{C'A'}{2}\sin\frac{B'D'}{2}+\sin\frac{A'B'}{2}\sin\frac{C'D'}{2}=0,$$

a factor $(\cot r_1 \cot r_2 \cot r_3 \cot r_4)^{\frac{1}{2}}$ being suppressed in each case.

In this case, the enveloping circle is small, and, in the touching four circles, $A'B'$, $A'C'$, ... denote common tangent arcs to pairs of circles.

33. Lastly, let one of the four circles become a point-circle, as at D'; then AD', BD', CD' denote tangent arcs drawn from a current point in the enveloping circle to the other three. Call these t_1, t_2, t_3.

Since S_1, S_2, S_3, of § 27, denote the equations to the triad, for an external point,

$$S_1 = \cos r_1 - \cos r_1 \cos t_1 = 2\cos r_1 \sin^2\frac{t_1}{2};$$

and S_2, S_3 have like values. Thus, from the theorem

$$\sin\frac{B'C'}{2}\sin\frac{t_1}{2}+\sin\frac{C'A'}{2}\sin\frac{t_2}{2}+\sin\frac{A'B'}{2}\sin\frac{t_3}{2}=0,$$

is deduced Dr. Casey's equation to a pair of enveloping circles,

$$\sin\frac{l}{2}\sqrt{(S_1\sec r_1)}+\sin\frac{m}{2}\sqrt{(S_2\sec r_2)}+\sin\frac{n}{2}\sqrt{(S_3\sec r_3)}=0.$$

Cor. As has been shown in § 30, this equation may take the form

$$\cos\frac{\phi_1}{2}\sqrt{(S_1\operatorname{cosec} r_1)}+\cos\frac{\phi_2}{2}\sqrt{(S_2\operatorname{cosec} r_2)}+\cos\frac{\phi_3}{2}\sqrt{(S_3\operatorname{cosec} r_3)}=0.$$

34. To deduce the equations to the inscribed and escribed circles in the triangle of reference and its associated triangles.

In these cases, $\pm S_1$ becomes

$$a\alpha+b\beta\cos c+c\gamma\cos b;$$

it denotes the side $B'C'$ of the polar triangle $A'B'C'$, and is expressed as one of its coordinates $(\alpha'=0)$, since

$$6V\alpha' = a\alpha+b\beta\cos c+c\gamma\cos b;$$

ϕ_1, ϕ_2, ϕ_3 become the angles of $A'B'C'$, and the required equations are

$$\cos\frac{A'}{2}\sqrt{(\pm\alpha')}+\cos\frac{B'}{2}\sqrt{(\pm\beta')}+\cos\frac{C'}{2}\sqrt{(\pm\gamma')}=0.$$

35. To deduce the equation to the circle described about the triangle of reference.

By the theorem of § 32,

$$\sin\frac{BC}{2}\sin\frac{AD}{2}+\sin\frac{CA}{2}\sin\frac{BD}{2}+\sin\frac{AB}{2}\sin\frac{CD}{2}=0,$$

this is the vector equation to the circumscribed circle.

But, by Todhunter's *Spher. Trig.*, p. 63,

$$\tan R\sin BC\sin CD\sin BCD=4\sin\frac{BD}{2}\sin\frac{CD}{2}\sin\frac{BC}{2},$$

and $$\sin CD\sin BCD=\sin\alpha.$$

By substituting in the preceding theorem, we derive the required equation $$\frac{1}{\sin\alpha}\tan\frac{BC}{2}+\frac{1}{\sin\beta}\tan\frac{CA}{2}+\frac{1}{\sin\gamma}\tan\frac{AB}{2}=0.$$

The vector-form of the equation to the circumscribed circle,

$$\sin\frac{a}{2}\sin\frac{\rho_1}{2}+\sin\frac{b}{2}\sin\frac{\rho_2}{2}+\sin\frac{c}{2}\sin\frac{\rho_3}{2}=0,$$

may be also thus reduced to the usual three-point equation. The vector-form may be expanded to

$$\sin^2 A\cos^2(S-A)(1-\cos\rho_1)^2+\dots$$
$$\dots-2\sin B\sin C\cos(S-B)\cos(S-C)(1-\cos\rho_2)(1-\cos\rho_3)-\dots=0,$$

since $$\sin^2\frac{a}{2}\sin B\sin C=-\cos S\cos(S-A),$$

and $$6V\cos\rho_1=a\alpha+b\beta\cos c+c\gamma\cos b.$$

This may be reduced to the complete square of the circle

$$\sin A\sin(S-A)\cos\rho_1+\sin B\sin(S-B)\cos\rho_2$$
$$+\sin C\sin(S-C)\cos\rho_3+\tfrac{1}{2}(6V)^2\sec S\left(\frac{\sin A}{\sin a}\right)^2=0.$$

Again, we find that

$$\sin A\sin(S-A)+\cos c\sin B\sin(S-B)+\cos b\sin C\sin(S-C)$$
$$=6V\frac{\sin A}{\sin a}\cot R=-\tfrac{1}{2}\sec S(\sin a\sin B\sin C)^2,$$

since the coordinates of the pole are

$$\alpha \operatorname{cosec}(S-A) = \beta \operatorname{cosec}(S-D) = \gamma \operatorname{cosec}(S-C) = \sin R.$$

This vector-form may be thus further reduced to the known forms

$$a\alpha + b\beta + c\gamma = 6V$$

and $$\frac{1}{\alpha}\tan\frac{a}{2} + \frac{1}{\beta}\tan\frac{b}{2} + \frac{1}{\gamma}\tan\frac{c}{2} = 0.$$

It may be remarked that this is the first and almost the sole application to Spherics, by Gudermann, of the three-point system, or, as he calls it, Plücker's system of coordinates, then (1830) recently introduced. (*Sphärik*, p. 160.)

For an associated triangle the vector equation of a circumscribed circle is $$\sin\frac{a}{2}\cos\frac{r_1}{2} + \cos\frac{b}{2}\sin\frac{r_2}{2} + \cos\frac{c}{2}\sin\frac{r_3}{2} = 0.$$

This may be reduced to the three-point equations

$$-a\alpha + b\beta + c\gamma = 6V$$

and $$-\frac{1}{\alpha}\tan\frac{a}{2} + \frac{1}{\beta}\cot\frac{b}{2} + \frac{1}{\gamma}\cot\frac{c}{2} = 0.$$

The limiting cases of the problem of contacts have thus been considered, when the radii become quadrants or zero.

36. Conversely, the equation to a pair of circles circumscribed about three others may be deduced from the equation to the circle inscribed in a triangle of reference.

The Fig. 5 is taken from Dr. Casey's memoir on the problem of contacts in planimetry (*Proc. of Royal Irish Academy*, Vol. IX.). But the circles must be considered as small spherical circles, and the sides of ABC as great circles. Three circles are drawn at the points of contact, of radii r_1, r_2, r_3. P is a point in the inscribed circle, radius R.

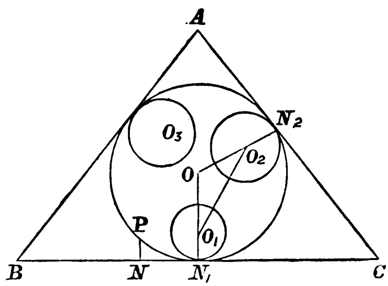


Fig. 5.

If S_1, S_2, S_3 denote, as in § 27, these three circles; then, if the coordinates be those of an external point P,

$$S_1 = \cos r_1 - \cos r_1 \cos t_1 = \cos r_1 - \cos PO_1$$

$$= \cos r_1 - \cos R \cos (R - r_1) - \sin R \sin (R - r_1) \cos POO_1$$

$$= 2 \sin R \sin (R - r_1) \sin^2 \tfrac{1}{2} (POO_1) = 2 \operatorname{cosec} R \sin (R - r_1) \sin^2 \tfrac{1}{2} (PN_1):$$

Hence $\qquad \sin \alpha = \sin PN_1 \cos PN_1 O = 2 \sin^2 \tfrac{1}{2} (PN_1) \cot R$

$$= S_1 \cos R \operatorname{cosec} (R - r_1).$$

Again, $\qquad \cos O_1 O_2 = \sin r_1 \sin r_2 + \cos r_1 \cos r_2 \cos n$

$$= \cos (r_1 - r_2) - 2 \cos r_1 \cos r_2 \sin^2 \frac{n}{2},$$

if, as in § 28, l, m, n denote common tangents to pairs of circles. Also

$$\cos O_1 O_2 = \cos (R - r_1) \cos (R - r_2) + \sin (R - r_1) \sin (R - r_2) \cos (O_1 O O_2)$$

$$= \cos (r_1 - r_2) - 2 \sin (R - r_1) \sin (R - r_2) \sin^2 \tfrac{1}{2} (O_1 O O_2),$$

and $\qquad \cos \frac{C}{2} = \cos R \sin \tfrac{1}{2} (O_1 O O_2).$

Therefore

$$\cos^2 \frac{C}{2} = \sin^2 \frac{n}{2} \cos^2 R \cos r_1 \cos r_2 \operatorname{cosec} (R - r_1) \operatorname{cosec} (R - r_2)$$

$$= \frac{1}{S_1 S_2} \sin \alpha \sin \beta \sin^2 \frac{n}{2} \cos r_1 \cos r_2.$$

Substitute for $\sin \alpha$, $\sin \beta$, $\sin \gamma$, in the equation to the inscribed circle,

$$\cos \frac{A}{2} \sqrt{(\sin \alpha)} + \cos \frac{B}{2} \sqrt{(\sin \beta)} + \cos \frac{C}{2} \sqrt{(\sin \gamma)} = 0.$$

The equation is deduced to a pair of circles circumscribed about three others,

$$\sin \frac{l}{2} \sqrt{(S_1 \sec r_1)} + \sin \frac{m}{2} \sqrt{(S_2 \sec r_2)} + \sin \frac{n}{2} \sqrt{(S_3 \sec r_3)} = 0.$$

37. The dirigent conic may be determined geometrically both in Spherics and in Planimetry.

The polar of a radical centre of three given circles with respect to

any one of them intersects their axis of similitude in a point, whose connector with the centre of that circle touches the dirigent conic in that point.

On referring to § 27, a radical centre is thus defined:—

$$(a\alpha+b\beta\cos c+c\gamma\cos b)\sec r_1 = (a\alpha\cos c+b\beta+c\gamma\cos a)\sec r_2$$
$$= (a\alpha\cos b+b\beta\cos a+c\gamma)\sec r_3.$$

The polar of this point (f, g, h) with respect to the circle S_1 is

$$(a\alpha+b\beta\cos c+c\gamma\cos b)(af+bg\cos c+ch\cos b)\sec^2 r_1$$
$$= a\alpha\,(af+bg\cos c+ch\cos b)+b\beta\,(af\cos c+bg+ch\cos a)$$
$$+c\gamma\,(af\cos b+bg\cos a+ch),$$

or $\quad (a\alpha+b\beta\cos c+c\gamma\cos b)\sec r_1 = a\alpha\cos r_1+b\beta\cos r_2+c\gamma\cos r_3.$

This may be simplified to the form (§ 30)

$$a\alpha\sin r_1-b\beta\sin r_2\cos\phi_3-c\gamma\sin r_3\cos\phi_2 = 0.$$

Through its intersection with the axis of similitude

$$a\alpha\sin r_1+b\beta\sin r_2+c\gamma\sin r_3 = 0,$$

the line $\qquad b\beta\sin r_2\sec^2\frac{\phi_2}{2}+c\gamma\sin r_3\sec^2\frac{\phi_3}{2} = 0$

passes, which is the tangent to the dirigent conic (§ 30)

$$\frac{1}{a\alpha}\cos^2\frac{\phi_1}{2}\operatorname{cosec} r_1+\frac{1}{b\beta}\cos^2\frac{\phi_2}{2}\operatorname{cosec} r_2+\frac{1}{c\gamma}\cos^2\frac{\phi_3}{2}\operatorname{cosec} r_3 = 0,$$

or $\quad \dfrac{1}{a\alpha}\sin^2\dfrac{l}{2}\sec r_1+\dfrac{1}{b\beta}\sin^2\dfrac{m}{2}\sec r_2+\dfrac{1}{c\gamma}\sin^2\dfrac{n}{2}\sec r_3 = 0.$

Three points A, B, C are therefore known, and the tangents at those points, of the dirigent conic; three other points can be found from the property given in § 5,

$$\cot\rho_1+\cot\rho_2 = 2\cot\rho.$$

And Pascal's theorem enables us to construct the conic, when five of its points are given.

38. Another solution of the problem of contacts is suggested by the perfect duality which exists in Spherical Geometry.

The dual forms of the equations (1) of § 27 denote in the tangential form, when they are referred to the polar triangle as the triangle of reference, three circles, which are polar or complementary to the three given circles S, S_2, S_3.

All three complementary circles have the type

$$ap - bq\cos C - cr\cos B = 6V\sin r_1.$$

An auxiliary Jacobian, the dual of the Jacobian in § 27,

$$ap\sin r_1 + bq\sin r_2 + cr\sin r_3 = 6V,$$

denotes a circle whose common tangents with these complementary circles are quadrants in length.

A pair of circles, which shall be circumscribed about these three circles, will be contangential with their Jacobian,—their common centre of similitude being a radical centre of the given circles.

The rest of the analytical solution would proceed as in § 27, *mutatis mutandis.*

39. The axes of similitude of the given circles S_1, S_2, S_3 in § 27 are quadrantal polars of the radical centres of their complementary circles in § 38.

This is evident both geometrically and analytically.

The line-equations of the primitive circles S_1, S_2, S_3 are

$$p^2 = \sin^2 r_1, \quad q^2 = \sin^2 r_2, \quad r^2 = \sin^2 r_3.$$

When combined, they denote the four axes of similitude. The three-point equivalent equations to their dual circles of § 38 are

$$\alpha'^2 = \cos^2 r_1', \quad \beta'^2 = \cos^2 r_2', \quad \gamma'^2 = \cos^2 r_3'.$$

When combined, they denote the four radical centres. The accents are added, to indicate reference to the polar triangle.

It is evident from a diagram that $p = \alpha'$, $q = \beta'$, $r = \gamma'$; and the radii of the triads of circles are complementary to each other.

40. The theorem here used may be generalised. Every tangential equation to a spherical curve referred to a triangle ABC has a corresponding three-point equation of the same form, referred to the polar triangle $A'B'C'$, to denote its complementary or polar curve.

[41. *Note on* §§ 23, 24.—In *plano* a circle is only dually inverted into another circle, when the angles of inversion are complementary.

$$\tan\frac{\rho_1}{2}\tan\frac{\rho_4}{2} = \tan\frac{\rho_2}{2}\tan\frac{\rho_3}{2} = 1, \text{ and } R = \frac{\pi}{2}.$$

Then the Boothian circle $\frac{c}{p} = 1 + m\xi + n\eta$ (ξ being the common radical axis) is transformed into $\frac{c}{p} = -1 - m\xi \pm n\eta$, and the common tangent of two circles is unaltered in length by Dual Inversion.]

Reciprocation in Statics. *By* Prof. Genese, M.A.

[*Read June* 10*th*, 1886.]

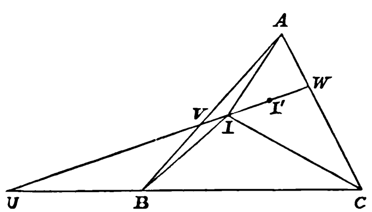
If a system of complanar forces P_1, P_2, &c., be in equilibrium, and any transversal meet their lines of action in I_1, I_2, &c., at angles θ_1, θ_2, &c. (estimated with due regard to the directions of the forces and one direction of the transversal), then, resolving perpendicularly to the transversal, we have a system of parallel forces $P_1 \sin\theta_1$, $P_2 \sin\theta_2$, &c., at I_1, I_2, &c., in equilibrium.

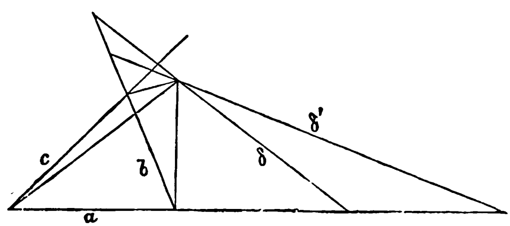
The opposite to one of the above forces is defined as the resultant of the others. The cases of four and five forces led to the theorems in the right-hand column of the following

Contrast.

1. If I be the centre of parallel forces l, m, n at the points A, B, C, then forces lIA, mIB, nIC are in equilibrium. (Leibnitz.)

1. If δ be the line of action of the resultant of forces l, m, n, acting along the sides a, b, c, then parallel forces $l\sin(\delta a)$, $m\sin(\delta b)$, $n\sin(\delta c)$ are in equilibrium.





2. Therefore, if x, y, z denote the connectors IA, IB, IC,

$$\frac{lIA}{\sin(yz)} = \frac{mIB}{\sin(zx)} = \frac{nIC}{\sin(xy)}.$$

2. Therefore, if X, Y, Z denote the intersections δa, δb, δc,

$$\frac{l\sin(\delta a)}{YZ} = \frac{m\sin(\delta b)}{ZX} = \frac{n\sin(\delta c)}{XY}.$$

3. If I' be a second point, the centre for forces l', m', n'; the *resultant* of forces $lI'A$, $mI'B$, $nI'C$ acts along $I'I$, and

$$= (l+m+n)\, I'I.$$

Similarly for forces $l'IA$, $m'IB$, $n'IC$.

3. If δ' be a second line for forces l', m', n'; the *centre* for parallel forces $l\sin(\delta' a)$, $m\sin(\delta'\beta)$, &c., is $\delta\delta'$, and resultant

$$= \{l, m, n\}\sin(\delta\delta'),$$

where $\{l, m, n\}$ = resultant of l, m, n. Similarly for parallel forces $l'\sin(\delta a)$, $m'\sin(\delta b)$, $n'\sin(\delta c)$.

4. The above six forces are in equilibrium—

if $\quad l+m+n = l'+m'+n'$, $\quad$ | if $\quad \{l, m, n\} = \{l', m', n'\}$.

These conditions are secured—

if l, m, n be the areal coordinates of I,

l', m', n' those of I'.

if l, m, n be the quantities ap, bq, cr, where p, q, r are the distances of δ from vertices of abc (for the resultant of these

$$= \sqrt{a^2p^2 + \ldots - 2bcqr\cos A - \&c.}$$
$$= 2\Delta),$$

l', m', n' the quantities for δ'.

5. If II' meet BC at U,

$$l : l' :: (UI) : (UI').$$

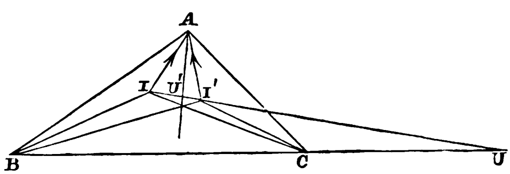
5. If $\delta\delta'$ connect with bc by u,

$$l : l' :: \sin(u\delta) : \sin(u\delta').$$

Whence, taking the six forces in pairs,* and observing that the three

* Let the resultant of $l'IA$ and $lI'A$ meet II' at U', then

$$U'I : U'I' :: \Delta U'IA : \Delta U'I'A.$$



And, by moments about U',

$$l'\Delta U'IA = l\Delta U'I'A',$$

therefore $\quad U'I : U'I' :: l : l' :: UI : UI'$,

or U' is harmonic conjugate of U with respect to II'.

resultants must meet in a point, the following theorems in Geometry may be obtained :—

If II' meet the sides of ABC in U, V, W, and U', V', W' be the harmonic conjugates of U, V, W with respect to I and I'; then AU', BV', CW' meet in one point.

If $\delta\delta'$ be joined to the vertices of abc by u, v, w, and u', v', w' be the harmonic conjugates of u, v, w with respect to δ and δ'; then au', bv', cw' are in one straight line.

An example of analytical correspondence is given by the following:

If parallel forces ω_1, ω_2, ω_3, ω_4, acting at points $(l_1 m_1 n_1)$, &c., be in equilibrium, and

If forces P_1, P_2, P_3, P_4, acting along lines $(l_1 m_1 n_1)$, &c., be in equilibrium, and

$$|\,1, 2, 3\,| \text{ denote } \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix},$$

then

$$\frac{\omega_1}{|\,2, 3, 4\,|} = \frac{-\omega_2}{|\,3, 4, 1\,|} = \text{\&c.},$$

or, ω_1, $-\omega_2$, &c. are proportional to the areas of the corresponding triangles with their proper signs.

then

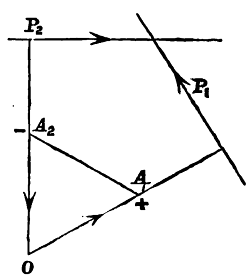
$$\frac{P_1}{|\,2, 3, 4\,|} = \frac{-P_2}{|\,3, 4, 1\,|} = \text{\&c.},$$

or P_1, $-P_2$, &c. are proportional to quotients of the form $\frac{\Delta}{R}$, where Δ is the area of the corresponding triangle, and R the radius of its circum-circle. (See Clifford's *Mathematical Papers*, p. 90.)

These theorems suggest the existence of a principle of reciprocation in Statics. This will now be presented synthetically.

It will be convenient henceforth to replace the term "parallel force" by "weight."

Let P_1, P_2, &c. be any system of complanar forces in equilibrium; A_1, A_2, &c. the poles of their lines of action with respect to any circle, centre O, radius unity. The forces P_1, P_2, &c., translated to O, would balance, and therefore also if rotated through $-90°$. They will then act along OA_1 OA_2, &c., from or towards O according to the sign of the moments of the forces about O.



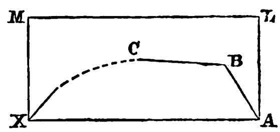
Hence, by Leibnitz' theorem, weights $\frac{P_1}{OA_1}$, $\frac{P_2}{OA_2}$, &c., at A_1, A_2, &c., are in equilibrium. That is, weights equal to

the moments of the forces P_1, P_2, &c., about O, placed at A_1, A_2, &c., balance. Reversing one of the forces, we have the resultant of the rest. Hence, *if weights be placed at the poles of the lines of action of any system of complanar forces with respect to a circle centre O, proportional to the moments of these forces about O (with their proper signs), the resultant force will act along the polar of the centre of the weights; and its magnitude and direction may be found from the resultant weight.*

Cor. 1. If the weights balance, the forces balance.

Cor. 2. If the weight centre be at O, the system of forces is equivalent to a couple, and conversely.

The following illustrates the theory.



Let $ABC \ldots X$ be any closed polygon, $ALMX$ a rectangle of double the area on the same side of AB. Let forces represented by AB, $BC \ldots$ act along all the sides except AX. Then AX is the magnitude of the resultant. By taking moments round A, it will be seen that LM represents completely the resultant. This is true when the polygon becomes a curve. Reciprocating, we obtain the weight-centre of the reciprocal curve properly loaded. In particular, if the curve be an arc of circle, centre O, weight-centre G, taking moments about O,

$$LM \times \frac{r^2}{OG} = \text{sum of moments round } O$$

$$= \text{twice sector } AOX,$$

$$= \text{arc } AX \times r,$$

whence $$OG = \frac{AX \times r}{\text{arc } AX}.$$

Again, *forces may be projected orthogonally according to the same laws as the lines representing them.* For, suppose a system of forces in equilibrium to be represented by straight lines, then their moments about any point are represented by areas. These areas project orthogonally in a constant ratio. The conditions for equilibrium are that the sums of moments round the vertices of a triangle should severally vanish; but the moments for the given system vanish, therefore they do for the projected system.

The same is true for weights. Therefore *the reciprocation theorem*

which has been proved for the circle will hold also for any ellipse; whence, also, using imaginary projection, for any hyperbola.

A corresponding theorem may be derived for the parabola; the centre being at infinity, the perpendiculars from it vary as the sines of the inclination of the forces to the axis of the parabola. Hence, *if forces* P_1, P_2, *&c. make angles* θ_1, θ_2, *&c. with one direction of the axis of a parabola, and weights* $P_1 \sin\theta_1$, $P_2 \sin\theta_2$, *&c. be placed at the poles of the lines of action with respect to the curve, then the resultant force acts along the polar of the weight-centre.*

The preceding theorems admit of easy verification by the methods of Cartesian coordinates.

The method of the paper has an interesting application to Kinetics. If G be the centre of inertia of a plane laminal mass m, at rest, and a force P begin to act in the plane at a distance $GL = p$ from G, it generates a linear acceleration $\dot{v}$ of the mass perpendicular to GL, and an angular acceleration $\dot{\omega}$ about G, given by

$$m\dot{v} = P \quad\dots\dots\dots\dots\dots\dots(1),$$

$$mk^2\dot{\omega} = Pp \quad\dots\dots\dots\dots\dots\dots(2),$$

where k^2 is the radius of gyration about G.

If I be the centre of instantaneous rotation, an angular velocity $\dot{\omega}\,dt$ must give velocity $\dot{v}\,dt$ to G, *i.e.*, $IG\,.\,\dot{\omega}$ must be equal to $\dot{v}$. Hence the known relation $$IG\,.\,p = k^2,$$
or $$GI\,.\,GL = -k^2,$$

i.e., I is the pole of the line of action of P with respect to a circle, centre G, radius $k\sqrt{-1}$.

Now, simultaneous small rotations may be compounded by the same rules as parallel forces, and for different forces are proportional to the moments Pp. Hence the following theorem:—

If any number of forces begin to act in the plane of a laminal mass, and weights, proportional to their moments about the centre of mass, be placed at the poles of the lines of action of the forces with respect to a circle about the centre of radius $\sqrt{-1}$ *times the radius of gyration round it;* then the centre of the weights is the centre of instantaneous rotation.*†

* Of course a real circle of radius k may be substituted, and the radius to the pole reversed.

† One of the referees points out that if the laminal mass be considered not at rest, but at any time during its motion, then the theorem gives the point which for the moment has no acceleration.

The following presents were received in the Recess :—

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A number of pamphlets by M. Émile Lemoine.

A number of pamphlets by M. Maurice D'Ocagne.

"A History of the Theory of Elasticity and of the Strength of Materials from

Galilei to the Present Time," by the late Isaac Todhunter, edited and completed for the Syndics of the University Press, by Karl Pearson. Galilei to Saint Venant, 1639—1850, Vol. I., pp. x., xvi., 924; Cambridge, 1886. From the Editor.

"Sur les Surfaces Anallagmatiques," par J. Neuberg (Association Française pour l'Avancement des Sciences, Congrès de Grenoble, 1885.)

"Sur quelques Systèmes de Tiges Articulées, tracé mécanique des Lignes," par J. Neuberg (Université de Liège, Association des élèves des écoles spéciales), 25 Av., 1885, et 21 Mars, 1886; Liège, 1886.

"Sur le Point de Steiner," par J. Neuberg (Journal de Mathématiques Spéciales); Paris, 1886.

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APPENDIX.

Mr. Basset's paper (p. 122) is published in the *Philosophical Magazine* for August, 1886 (Vol. XXII., No. 135, pp. 140—144).

Prof. Neuberg's article, "Sur Quelques Systèmes de Tiges Articulées" (above), contains a good deal of information on Linkages.

The short communication by Mr. H. M. Taylor (p. 127) is embodied in an article entitled, "On a Geometrical Interpretation of the Algebraical Expression which equated to zero represents a Curve or a Surface," in the *Messenger of Mathematics* for July, 1886 (Vol. XVI., No. CLXXXIII., pp. 39—41).

Mr. J. Griffiths sends the following Note:

"In a paper published in the *Proceedings* of the Society (Vol. XIV., p. 196), it was shown by me that the integral relation

$$t = p + q \operatorname{sn} u \operatorname{sn}(u - u_0)$$

gives a differential equation

$$\frac{dt}{\sqrt{T}} = M du, \quad \text{where} \quad T = (a_0, a_1, a_2, a_3, a_4)(t, 1)^4.$$

The object of the present note is to point out that the above leads to an easy method of finding the integral of

$$\frac{dx}{\sqrt{X}} \pm \frac{dy}{\sqrt{Y}} = 0,$$

where $X=(c_0, c_1, c_2, c_3, c_4)(x, 1)^4$ and $Y=(c_0, c_1, c_2, c_3, c_4)(y, 1)^4$.

In fact, since $t=p+q\ \mathrm{sn}\, u\ \mathrm{sn}\,(u-u_0)$ is equivalent to

$$t=p+q\left\{\frac{\mathrm{sn}^2\left(u-\frac{u_0}{2}\right)-\mathrm{sn}^2\frac{u_0}{2}}{1-k^2\,\mathrm{sn}^2\frac{u_0}{2}\,\mathrm{sn}^2\left(u-\frac{u_0}{2}\right)}\right\},$$

or
$$\mathrm{sn}^2\left(u-\frac{u_0}{2}\right)=\frac{q\,\mathrm{sn}^2\frac{u_0}{2}-p+t}{q-k^2p\,\mathrm{sn}^2\frac{u_0}{2}+k^2\,\mathrm{sn}^2\frac{u_0}{2}\,t},$$

it follows that if, in the integral addition equation

$$\{\xi^2+\eta^2-k^2\sin^2\mu\ \xi^2\eta^2-1-\cos^2\mu\}^2=4\cos^2\mu\ (1-\xi^2)(1-\eta^2),$$

we make the homographic substitutions

$$\xi^2=\frac{a+x}{b+cx},\quad \eta^2=\frac{a+y}{b+cy},$$

there results an integral relation between x and y which must lead to a differential equation of the form

$$\frac{dx}{\sqrt{X}}\pm\frac{dy}{\sqrt{Y}}=0.$$

There is no difficulty in working out the details.

The above addition equation in ξ, η, where μ is an arbitrary constant, gives
$$\frac{d\xi}{\sqrt{1-\xi^2\,.\,1-k^2\xi^2}}\pm\frac{d\eta}{\sqrt{1-\eta^2\,.\,1-k^2\eta^2}}=0,$$

and
$$\xi^2=\frac{a+x}{b+cx},\quad \eta^2=\frac{a+y}{b+cy}$$

give
$$d\xi=\frac{b-ac}{2}\ \frac{dx}{(a+x)^{\frac{1}{2}}(b+cx)^{\frac{3}{2}}},$$

$$\sqrt{1-\xi^2\,.\,1-k^2\xi^2}=\frac{\sqrt{b-a+(c-1)\,x\,.\,b-k^2a+(c-k^2)\,x}}{b+cx},$$

with similar expressions with respect to η and y. Hence

$$\frac{dx}{\sqrt{X}}\pm\frac{dy}{\sqrt{Y}}=0;$$

where $X = a+x \,.\, b+cx \,.\, b-a+(c-1)\,x \,.\, b-k^2a+(c-k^2)\,x,$

$$Y = a+y \,.\, b+cy \,.\, b-a+(c-1)\,y \,.\, b-k^2a+(c-k^2)\,y,$$

and a, b, c, k are four independent constants, say given constants.

Conversely, the complete integral of

$$\frac{dx}{\sqrt{X}} \pm \frac{dy}{\sqrt{Y}} = 0$$

is $\quad (\xi^2+\eta^2-k^2\sin^2\mu\,\xi^2\eta^2-1-\cos^2\mu)^2 = 4\cos^2\mu\,(1-\xi^2)(1-\eta^2),$

where $\quad \xi^2 = \dfrac{a+x}{b+cx}, \quad \eta^2 = \dfrac{a+y}{b+cy},$

and μ is the arbitrary constant of integration.

[For a discussion of the differential equation

$$\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}},$$

see Chap. XIV. of Prof. Cayley's 'Elliptic Functions.']."

We are informed that reference is made to Mr. Kempe's remarks on Map colouring (*Proc.*, Vol. X., p. 229) in a recent number of the "Berichte über die Verhandlungen der K. Sächsischen Gesellschaft der Wissenschaften zu Leipzig."

For an abstract of Mr. Kempe's remarks (p. 196), see *Messenger of Mathematics*, Vol. XV., p. 188—*cf.*, also his "Memoir on the Theory of Mathematical Form" (*Phil. Trans.*, Pt. i., 1886.)

The method of Mr. Griffiths referred to by Mr. Leudesdorf, in § 15 of his paper (p. 210), was contained in a paper read before the Society on March 11th, 1886, entitled, "On the Invariantisers of a Binary Quantic." The scope of this paper may be briefly described as follows:—

Taking an invariant to be a function of the coefficients $a_0, a_1, \dots a_n$ of a binary quantic,

$$I_n(a_0, a_1, \dots a_n, x, y) = a_0x^n + na_1x^{n-1}y + \frac{n(n-1)}{1\,.\,2}a_2x^{n-2}y^2 + \dots + a_ny^n,$$

which is reduced to zero by one, or both, of the operators

$$\Omega = a_0\frac{d}{da_1} + 2a_1\frac{d}{da_2} + \dots + na_{n-1}\frac{d}{da_n},$$

$$O = a_n\frac{d}{da_{n-1}} + 2a_{n-1}\frac{d}{da_{n-2}} + \dots + na_1\frac{d}{da_0},$$

then such functions of the elements $a_0, a_1, \dots a_n$ as will, when substi-

tuted for x and y respectively in I_n, make the quantic an invariant, are called *Invariantisers* of the quantic.

One of the fundamental theorems is

$$\Omega I_n(a_0, a_1, \ldots a_n, x, y) = \frac{dI_n}{dx}(\Omega x + y) + \frac{dI_n}{dy}\Omega y;$$

$$OI_n(a_0, a_1, \ldots a_n, x, y) = \frac{dI_n}{dx}Ox + \frac{dI_n}{dy}(Oy + x).$$

Hence it appears, for example, that if $\Omega y = 0$ and $\Omega x + y = 0$, then I_n is annihilated by Ω; *i.e.*, I_n is an invariant according to the above definition. For instance, if we take as invariantisers $y = a_0$ and $x = -a_1$, we have

$$I_1 = a_0 x + a_1 y = 0,$$

$$I_2 = a_0 x^2 + 2a_1 xy + a_2 y^2 = a_0(a_0 a_2 - a_1^2),$$

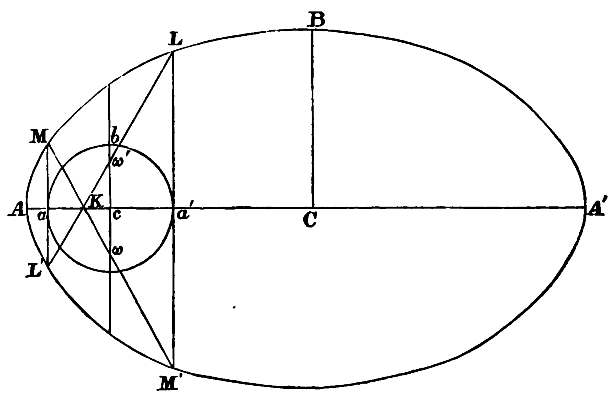
$$I_3 = a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3 = a_0(2a_1^3 - 3a_0 a_1 a_2 + a_0^2 a_3),$$

and so on.

The following is the short Note by Mr. Simmons (p. 197),* entitled

A Theorem in Conics.

Through the focus K of an ellipse draw LKL', MKM' chords



* With respect to this Prof. Neuberg writes,—"Elle peut être le point de départ de nouveaux développements sur cette géométrie Brocardienne."

making an angle of 60° with the major axis; join LM', ML' and describe an ellipse, focus K and LM', ML' for tangents at vertices, then

(i.) The eccentricity of the inner ellipse aba' is half that of the outer ellipse ABA'.

(ii.) They both have the same directrix nearer K.

(iii.) An infinite number of triangles can be inscribed in one which are circumscribed to the other.

(iv.) If ABA' be orthogonally projected into a circle, the triangles all have the projection of K for Symmedian point, the ellipse aba' becomes the Brocard-ellipse, the intersections of LL' and MM' with bc become the Brocard-points, the Brocard-angle $= \sin^{-1}\left(\frac{bc}{BC}\right)$, &c., &c.

Or, conversely :—If ω, ω' be Brocard-points, and K Symmedian point of a triangle, and if angles $K\omega\omega'$, $K\omega'\omega$ be each orthogonally projected into angles of 60°, the circum-circle and Brocard ellipse project into ellipses with the same focus.

We have not been able to get much information about Mr. Duncan Brockelbank, late Assistant Actuary, Continental Life Insurance Company, New York. From the *Times* we gather that he was the third son of the late Lemuel Brockelbank, M.A., formerly of Greenwich. Mr. Woolhouse, who was slightly acquainted with him, informs us, on his grandson's authority, that "he was of very active habits, and clever as an accountant and business man; he brought out the scheme for the 'Times' Assurance Company, Limited, which embodied several new ideas. The Company *nearly* got a start." Mr. Brockelbank died September 27th, 1885, on board the "Melbourne," during the passage to Australia, where he had intended to reside. He was in his 36th year.

The following corrections should be made in the present volume :—

p. 109, first line of § 4, *read*,—"we have, if $(u)_n \equiv \alpha_0 U + \alpha_1 U' = \alpha U - \alpha' U'$, &c."

p. 256, *add* $= 0$ in equation (1).

p. 257, equation (8), *for* $b_3\Omega$ *read* $b\Omega_3$.

p. 259, the second equation should have the additional terms

$$\frac{\dot{a}^2}{a^2} - \frac{w-w'}{a}\left\{(a-b)\,w-(a+b)\,w'\right\} + \frac{v-v'}{a}\left\{(c-a)\,v+(c+a)\,v'\right\}.$$

p. 211, last line but one of § 15, reference should be to "§ 11" *instead of* "§ 12."

The following results were arrived at in attempting to generalize known properties of the Brocard and Lemoine points.

I. A predominating constant is $K = a^2+b^2+c^2$; if we take a point P, such that

$$-\alpha_1 = 2a\Delta/(b^2+c^2-a^2),\ \beta_1 = 2b\Delta/(b^2+c^2-a^2),\ \gamma_1 = 2c\Delta/(b^2+c^2-a^2),$$

we have $-\alpha_1 = a \tan A/2,\ \beta_1 = b \tan A/2,\ \gamma_1 = c \tan A/2,$

and, O being circumcentre, we find

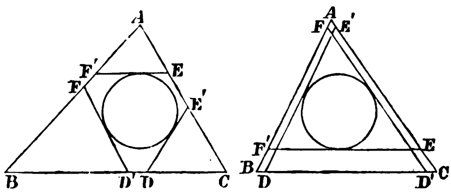
$$OP_1 = R \sec A, \quad OP_2 = R \sec B, \quad OP_3 = R \sec C,$$

and therefore $BOCP_1$, &c. are concyclic, or OCP_1, &c. are right angles, hence AP_1, BP_2, CP_3 are Symmedian lines. These points are then Neuberg's "Associés du point de Lemoine" (*Mathesis* I., 1881). I have not met with the equation to the circle $P_1P_2P_3$. I find it to be

$$4\,(\alpha\beta\gamma+\ldots)+(a \sec B \sec C+\ldots)(a\alpha+\ldots) = 0,$$

which shows that it is coaxal with the circum-, polar-, nine-point, and orthocentroidal circles [see below p. 425 (xii.)].

II. I next examined what would be the result if the "Lemoine" point were enlarged so as to be circular (and triangular); *i.e.*, for what circles the intersections of tangents to them parallel to the sides of the triangles would give concyclic points. The results, obtained mostly in November, 1886, follow.*



For simplicity, let (O) be a circle entirely within the triangle ABC to which are drawn the two sets of tangents parallel to the sides, and cutting them in the points D, D'; E, E'; F, F' (see figures).

Let the coordinates of O be (h, k, l) and the radius equal r', then

$$\left.\begin{aligned} h\pm r' &= BF' \sin B = CE \sin C \\ k\pm r' &= CD' \sin C = AF \sin A \\ l\pm r' &= AE' \sin A = BD \sin B \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)}.$$

* I am well acquainted with the ordinary treatment of the circles, but the method here adopted led to the discovery of the conjugate circles.

Again,
$$\left.\begin{array}{l} \left.\begin{array}{r} AF'\sin B = c\sin B-(h\pm r') \\ = b\sin C-(h\pm r') \end{array}\right\} = AE\sin C \\ BD'\sin C = BF\sin A = a\sin C-(k\pm r') \\ CE'\sin A = CD\sin B = b\sin A-(l\pm r') \end{array}\right\} \quad \text{.........(ii.).}$$

If, now, the respective sets of six points are concyclic, we have, from (i.) and (ii.),

$$\frac{h\pm r'}{a} = \frac{k\pm r'}{b} = \frac{l\pm r'}{c} = \frac{2\Delta\pm 2r's}{a^2+b^2+c^2} = \frac{2(\Delta\pm r's)}{K} \quad \text{.........(iii.),}$$

whence it follows that the locus of O is the straight line

$$(b-c)\,h+(c-a)\,k+(a-b)\,l = 0 \quad \text{...............(iv.);}$$

hence all the centres lie on the line joining the Symmedian point (K) to the in-centre of ABC. Also, we learn from (iii.) that, if O_1, O_2 be centres of equiradial circles of the two systems,* then

$$KO_1 = KO_2 \quad \text{..............................(v.),}$$

or we may say that the distance of the Symmedian point from the in-centre is the A. M. between the distances of the *auxiliary* centres from the same point.

From (i.) and (iii.), we have

$$BF' : CD' : AE' = ab^{-1} : bc^{-1} : ca^{-1},$$

i.e., $$BF' = pa/b,\quad CD' = pb/c,\quad AE' = pc/a \quad \text{............(vi.),}$$

or the series of circles $DE'EF'FD'$ are "Tucker" circles.†

In the notation of (I),

$$t = \frac{h\pm r'}{a}\cdot\frac{1}{2\Delta} = \frac{1}{K}\pm\frac{r's}{K\Delta} = \frac{1}{K}\pm\frac{1}{K}\cdot\frac{r'}{r} \quad \text{.........(vii.),}$$

and therefore $tK-1 = \pm r'/r$, where r is the in-radius of ABC, and

$$\sigma^2 = t^2\lambda^2\mp r'/r = t^2\lambda^2\mp q'r' \text{.....................(viii.).}$$

Let t_1, t_1' be the values of t, and ρ_1, ρ_1' the radii of the two "Tucker" circles‡ (corresponding to the upper and lower sign respectively of the ambiguity), then we have

$$(t_1+t_1')\,K = 2,\quad (t_1-t_1')\,K = 2q'r' \quad \text{...............(ix.),}$$

* *Auxiliary* circles to the group.

† See "A Group of Circles," *Quar. Jour. of Math.*, Vol. xx., No. 77, cited as (I.).

‡ These we propose to call *conjugate* Tucker circles. The director triangles are directly and inversely homothetic with ABC, K being the centre.

$$\left.\begin{aligned}\frac{\rho_1^2}{R^2} &= t_1^2\lambda^2 - q'r' \\ \frac{\rho_1'^2}{R^2} &= t_1'^2\lambda^2 + q'r'\end{aligned}\right\}, \quad \left.\begin{aligned}\text{therefore } \lambda^2(t_1^2+t_1'^2) &= (\rho_1^2+\rho_1'^2)/R^2 \\ K^2(t_1^2+t_1'^2) &= 2(r^2+r'^2)/r^2\end{aligned}\right\} \;\ldots\;(\text{x.}).$$

Hence $$2\mu^2R^2/r^2 = (\rho_1^2+\rho_1'^2)/(r^2+r'^2),$$

i.e., if ρ be the "T. R." radius, and ρ' the "Br." radius,

$$\left.\begin{aligned}\frac{2\rho^2}{r^2} = \frac{\rho_1^2+\rho_1'^2}{r^2+r'^2} &= \frac{\rho_1^2+\rho_1'^2-2\rho^2}{r'^2} \\ \rho_1'^2-\rho_1^2 &= (\rho'^2+\rho^2)\,2r'/r^*\end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{xi.}).$$

From [I. (vi.)] we get the sum of the perimeters of a pair of conjugate "Tucker" hexagons to be, by (ix.),

$$= 3\,(t_1+t_1')\,abc + a\left[1+t_1a^2-t_1K+1+t_1'a^2-t_1'K\right]+\ldots+\ldots$$

$$= \frac{2}{K}\left[3abc+a^3+b^3+c^3\right] = 2 \text{ perimeter of "Lemoine" hexagon,}$$

or the sums of conjugate "T." hexagons are isoperimetrical; it is seen, also, that the sums of the corresponding sides of the conjugates equal twice the corresponding sides of the "Lemoine" hexagon.

Again, the sum of the areas of the conjugate "T." hexagons

$$= 2\Delta\left[1-t_1^2\lambda^2+1-t_1'^2\lambda^2\right] = 2\Delta\left[2-(\rho_1^2+\rho_1'^2)/R^2\right]\ldots\ldots(\text{xii.}).$$

From (viii.) of (I.), and (vii.) above, the α coordinate of the mid-point of the join of the two conjugate centres is

$$R\left[\cos A+(Ka^2-\nu^4)/2bcK\right]$$
$$= R\left[K(b^2+c^2)-\nu^4\right]/2bcK$$
$$= R\,(bc+a^2\cos A)/K,$$

hence this mid-point coincides with the "T.R." centre. Assume

$$D'F' = qa, \quad \angle BD'F' = \theta,$$

then $$\sin\theta = (h+r')/qa = 2\Delta\,(1+q'r')/qK,$$

and $$a-(K+r')/\sin C = BD' = (h+r')\cot B + qa\cos\theta,$$

i.e., $$a - qb\sin\theta/\sin C = qa\,(\sin\theta\cot B+\cos\theta)$$
$$= qa\sin(\theta+B)/\sin B;$$

* See "The Triplicate-ratio Circle," *Quar. Jour.*, Vol. XIX., No. 76, p. 345, cited as (II.)

therefore

$$a \sin B \sin C = q \left[(a \sin C \cos B + b \sin B) \sin \theta + a \sin C \sin B \cos \theta \right],$$

i.e., $1/q \sin \theta = \cot \omega + \cot \theta = K/2\Delta\,(1+q'r')$
$= 2 \cot \omega/(1+q'r')$, by (II., xv.);

$$\left.\begin{array}{ll} \text{hence} & \dfrac{\cot \omega - \cot \theta}{\cot \omega + \cot \theta} = q'r' \\ \text{and} & \cot \theta / \cot \omega = (\Delta - r's)/(\Delta + r's) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots \text{(xiii.)}.$$

This result, bearing in mind (vii.) above, is identical with (I. vii.).

Referring to (I. ix.) for the equations to the "T." circles, we have, bearing in mind (ix.) for conjugates, the equation to the radical axis of such a pair to be

$$b^2c^2 \cos A\alpha + c^2a^2 \cos B\beta + a^2b^2 \cos C\gamma = 0 \ldots\ldots\ldots\ldots \text{(xiv.)},$$

or pairs of conjugate circles have a common radical axis, which is, of course, parallel to the radical axis of the "T.R." and circum-circles, and cuts circum-Brocard axis in the point

$$a\,(16\Delta^2 - a^2K) : b\,(16\Delta^2 - b^2K) : c\,(16\Delta^2 - c^2K).$$

We also get results (geometrically) which are obtained in (xi.), and below on the same page.

Considering any pair of "Tucker" circles, we get the equation to the radical axis to be

$$bc\alpha + ca\beta + ab\gamma = abc\,(a\alpha + b\beta + c\gamma)(t+t') \ldots\ldots\ldots\ldots \text{(xv.)},$$

and, combining this with the equation to the circles, we have

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = abc\,tt'\,(a\alpha + b\beta + c\gamma)^2 \ldots\ldots\ldots\ldots \text{(xvi.)}.$$

Hence, if $tt' = T/K^2$, with $t/K = 1 + q'r'$, $t'/K = 1 - q'r''$, we get that the intersections of this series of pairs of "T." circles lie on the concentric circles $K^2\,(a\beta\gamma + \ldots + \ldots) = abcT\,(a\alpha + \ldots + \ldots)^2$,

given by assigning different values to T.

III. In this section I give results arrived at in December, 1886, and January, 1887; most of them are well-known, but the methods are somewhat different from those employed elsewhere, and the properties of the orthocentroidal circle are in many cases, I believe, novel.

Let ΩAB, $\Omega' AC$ be the Brocard angles of a triangle ABC, then the following equations hold for the circles through (B, C), (C, A),

(A, B), respectively :—

$$\left.\begin{array}{lll}(\Omega) & c\,(a\beta\gamma+\ldots)-ba\,(a\alpha+\ldots)=0 & (S_1)\\ (\Omega') & b\,(a\beta\gamma+\ldots)-ca\,(a\alpha+\ldots)=0 & (\sigma_1)\\ (\Omega) & a\,(a\beta\gamma+\ldots)-c\beta\,(a\alpha+\ldots)=0 & (S_2)\\ (\Omega') & c\,(a\beta\gamma+\ldots)-a\beta\,(a\alpha+\ldots)=0 & (\sigma_2)\\ (\Omega) & b\,(a\beta\gamma+\ldots)-a\gamma\,(a\alpha+\ldots)=0 & (S_3)\\ (\Omega') & a\,(a\beta\gamma+\ldots)-b\gamma\,(a\alpha+\ldots)=0 & (\sigma_3)\end{array}\right\}\ldots\ldots\ldots(\text{i.}).$$

From these equations it is evident that the radical axes of $S_1\sigma_2$, $S_2\sigma_3$, $S_3\sigma_1$ cointersect in the Symmedian point (K), and those of $S_1\sigma_3$, $S_2\sigma_1$, $S_3\sigma_2$ in the centroid (G); and if G lies on σ_1, (S_2), then K lies on S_1, (σ_2), and so on. ....(ii.).

Again, $S_2\sigma_3$ intersect in p ($2bc\cos A/a, b, c$), $S_3\sigma_1$ in q ($a, 2ac\cos B/b, c$), $S_1\sigma_2$ in r ($a, b, 2ab\cos C/c$), where p, q, r are the angular points of Brocard's second triangle. ....(iii.).

Again, $\sigma_2 S_3$ intersect in p' ($a/2bc\cos A, 1/b, 1/c$), $\sigma_3 S_1$ in q' ($1/a, b/2ca\cos B, 1/c$), and $\sigma_1 S_2$ in r' ($1/a, 1/b, c/2ab\cos c$). ....(iv.).*

The radical axis of S_1S_2 is $ab\alpha = c^2\beta$, of S_2S_3 is $bc\beta = a^2\gamma$, and of S_3S_1 is $ca\gamma = b^2\alpha$ ....(v.),

in like manner, the radical axis of $\sigma_1\sigma_2$ is $ab\beta=c^2\alpha$, of $\sigma_2\sigma_3$ is $bc\gamma=a^2\beta$, and of $\sigma_3\sigma_1$ is $ca\alpha = b^2\gamma$ ....(vi.);

the radical centre of

$$\left.\begin{array}{lll}S_1, \sigma_2, \sigma_3\ (p_1) & \text{is} & ac,\ bc,\ a^2\\ S_2, \sigma_3, \sigma_1\ (p_2) & \text{is} & b^2,\ ab,\ ac\\ S_3, \sigma_1, \sigma_2\ (p_3) & \text{is} & ab,\ c^2,\ bc\end{array}\right\}\ldots\ldots\ldots\ldots(\text{vii.});$$

of

$$\left.\begin{array}{lll}\sigma_1, S_2, S_3\ (\pi_1) & \text{is} & ab,\ a^2,\ bc\\ \sigma_2, S_3, S_1\ (\pi_2) & \text{is} & ac,\ bc,\ b^2\\ \sigma_3, S_1, S_2\ (\pi_3) & \text{is} & c^2,\ ab,\ ac\end{array}\right\}\ldots\ldots\ldots\ldots(\text{viii.}).$$

If we denote the Brocard circle by B, then the radical centre of

$$\left.\begin{array}{lll}\sigma_2, S_3, B\ (\beta_1) & \text{is} & a^3,\ bc^2,\ b^2c\\ \sigma_3, S_1, B\ (\beta_2) & \text{is} & ac^2,\ b^3,\ a^2c\\ \sigma_1, S_2, B\ (\beta_3) & \text{is} & ab^2,\ a^2b,\ c^3\end{array}\right\}\ldots\ldots\ldots\ldots(\text{ix.}).$$

* The points (iii.), (iv.) are evidently what Prof. Neuberg calls isogonal conjugate points.

The line joining Ω (ac^2, ba^2, cb^2) to p (iii.) is

$$abc\alpha - c(b^2+c^2)\beta + a^2b\gamma = 0,$$

and that joining Ω' (ab^2, bc^2, ca^2) to q is

$$-c(c^2+a^2)\alpha + abc\beta + ab^2\gamma = 0;$$

these intersect in (ab^2, a^2b, c^3), *i.e.* (ix.) in the point β_3 [in the case supposed in (ii.) this point lies on the circum-Brocard axis].

In like manner,

$$\Omega q\ \left[b^2c\alpha + abc\beta - a(c^2+a^2)\gamma = 0\right]$$

meets $\quad \Omega' r\ \left[bc^2\alpha - a(a^2+b^2)\beta + abc\gamma = 0\right]$

in a^3, bc^2, b^2c, *i.e.*, in β_1; and Ωr, $\Omega' p$ meet in β_2.

Again, Ωr, $\Omega' q$; Ωp, $\Omega' r$; Ωq, $\Omega' r$ meet BC, CA, AB where they are met by $bc\alpha + ca\beta + ab\gamma = 0$.

The equation to the circle round $p'q'r'$ being written in the form

$$a\beta\gamma + \ldots = (a\alpha + \ldots)(\lambda\alpha + \mu\beta + \nu\gamma),$$

we have to determine λ from the equation

$$\lambda \begin{vmatrix} 2b\cos C, & 2a\cos C, & c \\ 2c\cos B, & b & 2a\cos B \\ a, & 2c\cos A, & 2b\cos A \end{vmatrix} = \begin{vmatrix} 2c\cos C, & 2a\cos C, & c \\ 2b\cos B, & b, & 2a\cos B \\ 2a\cos A, & 2c\cos A, & 2b\cos A \end{vmatrix},$$

i.e., $3\lambda(1-8\cos A\cos B\cos C) = 2\cos A(1-8\cos A\cos B\cos C)$,

therefore $\quad 3\lambda = 2\cos A,$

and the equation required is

$$3(a\beta\gamma + \ldots) = 2(a\alpha + \ldots)(\alpha\cos A + \beta\cos B + \gamma\cos C) \ldots\ldots (\text{x.});$$

this evidently passes through the centroid and through the orthocentre, and may hence be called the "orthocentroidal" circle. It has for its diameter the join of the last named points ....(xi.)*

It is evident, from (x.), that the orth.-, circum-, nine-point, and polar circles have a common radical axis, viz.,

$$\alpha\cos A + \beta\cos B + \gamma\cos C = 0 \ldots\ldots\ldots\ldots\ldots\ldots (\text{xii.}).$$

The radical axis of the "B." and orth. circles has for its equation

$$3(bc\alpha + \ldots) = 2K(\alpha\cos A + \ldots),$$

---

* This equation has also been obtained by Captain Brocard under the form

$$a^2\sin 2A + \ldots + \ldots = \alpha\beta\sin C + \ldots + \ldots$$

See Casey, Conics: *cf.*, also *Reprint*, Vol. XLIV., p. 111; and Artzt, *Programm*, 1886, p. 19.

which is satisfied by

$$a\,(b^2+c^2),\quad b\,(c^2+a^2),\quad c\,(a^2+b^2),$$

hence it passes through the mid-point of $\Omega\Omega'$.

Denoting the circum-, orth.-, and orthocentres by the letters O, O', and H, respectively, we see, since $3OG = OH$, that, from (xi.), the orth. circle is seen from O under an angle of 60°.

The polar of O with respect to O' has for its equation

$$\alpha \sin 3A+\beta \sin 3B+\gamma \sin 3C = 0,$$

which passes through the nine-point centre, as from the last paragraph is evident geometrically, and (tangent from $O)^2 = 3\rho^2$, where ρ is the orth. radius.

The polar of the Symmedian point is

$$\left\{\cos (B-C)+\cos 3A\right\}\alpha+\ldots+\ldots = 0 \quad\ldots\ldots\ldots\ldots(\text{xiii.}),$$

and the tangent through the centroid has for equation

$$(a^3-2abc\cos A)\,\alpha+\ldots+\ldots = 0 \quad\ldots\ldots\ldots\ldots\ldots(\text{xiv.}).$$

The polars of Ω, Ω' are

$$\left.\begin{array}{l} bc\,(a^2+b^2)\,\alpha+ca\,(b^2+c^2)\,\beta+ab\,(c^2+a^2)\,\gamma = 0 \\ bc\,(a^2+c^2)\,\alpha+ca\,(b^2+a^2)\,\beta+ab\,(c^2+b^2)\,\gamma = 0 \end{array}\right\}\ldots\ldots(\text{xv.});$$

these intersect on

$$bc\,(b^2-c^2)\,\alpha+\ldots+\ldots = 0,$$

i.e., on the circum-B. axis.

The equations to p_1p_2, p_2p_3, p_3p_1 are

$$\left.\begin{array}{l} ab\,(c^2-a^2)\,\alpha+a^2\,(b^2-c^2)\,\beta+bc\,(a^2-b^2)\,\gamma = 0 \\ ca\,(b^2-c^2)\,\alpha+bc\,(a^2-b^2)\,\beta+b^2\,(c^2-a^2)\,\gamma = 0 \\ c^2\,(a^2-b^2)\,\alpha+ab\,(c^2-a^2)\,\beta+ca\,(b^2-c^2)\,\gamma = 0 \end{array}\right\}\ldots\ldots(\text{xvi.}).$$

The triangles $p_1p_2p_3$ and ABC are triply in perspective; the axes of perspective, respectively, are

$$\left.\begin{array}{ll} \text{for } K, & a\,(b^2-c^2)(c^2-a^2)\,\alpha+b\,(c^2-a^2)(a^2-b^2)\,\beta \\ & \qquad +c\,(a^2-b^2)(b^2-c^2)\,\gamma = 0 \\ \text{for } G, & c^2a\,(a^2-b^2)(b^2-c^2)\,\alpha+a^2b\,(b^2-c^2)(c^2-a^2)\,\beta+\ldots = 0 \\ \text{for } \Omega', & bc\,(c^2-a^2)(a^2-b^2)\,\alpha+ca\,(a^2-b^2)(b^2-c^2)\,\beta+\ldots = 0 \end{array}\right\}\ldots(\text{xvii.}).$$

The triangles $\pi_1\pi_2\pi_3$ and ABC are in perspective with regard to G, and the equations to $\pi_1\pi_2$, $\pi_2\pi_3$, $\pi_3\pi_1$ and the axis of perspective are, respectively,

$$\left.\begin{aligned} b^2(c^2-a^2)\,\alpha+ab\,(b^2-c^2)\,\beta+ca\,(a^2-b^2)\,\gamma &= 0\\ ab\,(b^2-c^2)\,\alpha+c^2\,(a^2-b^2)\,\beta+bc\,(c^2-a^2)\,\gamma &= 0\\ ca\,(a^2-b^2)\,\alpha+bc\,(c^2-a^2)\,\beta+a^2\,(b^2-c^2)\,\gamma &= 0\\ ab^2(b^2-c^2)(c^2-a^2)\,\alpha+bc^2(c^2-a^2)(a^2-b^2)\,\beta+\ldots &= 0\end{aligned}\right\}\ldots\text{(xviii.)}.$$

Again, the triangles $\beta_1\beta_2\beta_3$, ABC are in perspective, with G as a centre of perspective; the equations to the sides and axis of perspective are

$$\left.\begin{aligned} bc\,(a^2c^2-b^4)\,\alpha+ca\,(b^2c^2-a^4)\,\beta+ab\,(a^2b^2-c^4)\,\gamma &= 0\\ bc\,(b^2c^2-a^4)\,\alpha+ca\,(a^2b^2-c^4)\,\beta+ab\,(c^2a^2-b^4)\,\gamma &= 0\\ bc\,(a^2b^2-c^4)\,\alpha+ca\,(a^2c^2-b^4)\,\beta+ab\,(b^2c^2-a^4)\,\gamma &= 0\\ bc\,(a^2b^2-c^4)(c^2a^2-b^4)\,\alpha+\ldots\qquad\ldots\qquad\ldots &= 0\end{aligned}\right\}\ldots\text{(xix.)}.$$

The triangles $p'q'r'$, ABC are in perspective, with G for centre of perspective; the equations in this case are exceedingly involved.

To return to the orth. circle, we obtain the polars of A, B, C to be

$$\left.\begin{aligned} -4a\cos A\alpha+c\beta+b\gamma &= 0\\ c\alpha-4b\cos B\beta+a\gamma &= 0\\ b\alpha+a\beta-4c\cos C\gamma &= 0\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(xx.)};$$

these lines meet the opposite sides of ABC in the points where the harmonic conjugates of the respective Symmedians, with regard to the sides, issuing from A, B, C, meet BC, CA, AB, and the area of the triangle formed by their intersection $= 12R^2\mu$, where

$$\mu = 1-8\cos A\cos B\cos C.$$

The polars for the points $(1, 1, 1)$, $(1, 1, -1)$, $(1, -1, 1)$, $(-1, 1, 1)$ are

$$\left.\begin{aligned} &\alpha\left[b+c-4a\cos A\right]+\ldots &&= 0\\ &\alpha\left[c-b-4a\cos A\right]+\beta\left[c-a-4b\cos B\right]+\gamma\left[a+b+4c\cos C\right] &&= 0\\ &\alpha\left[b-c-4a\cos A\right]+\beta\left[a+c+4b\cos B\right]+\gamma\left[b-a-4c\cos C\right] &&= 0\\ &\alpha\left[b+c+4a\cos A\right]+\beta\left[a-c-4b\cos B\right]+\gamma\left[a-b-4c\cos C\right] &&= 0\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\text{(xxi.)}.$$

The last three lines, of course, intersect in pairs on the lines (xx.).

Referring (O') to rectangular axes through O, parallel and perpendicular to BC, and assuming that OH makes an angle θ with BC, we have, for equation to (O'),

$$(x-2\rho\cos\theta)^2+(y-2\rho\sin\theta)^2=\rho^2,$$

or $$x^2+y^2-4\rho\cos\theta x-4\rho\sin\theta y+3\rho^2=0 \text{...........(xxii.)}.$$

We have, for determining the envelope of the polar of any point on the circum-circle with respect to (O'),

$$x'(x-2\rho\cos\theta)+y'(y-2\rho\sin\theta)=2\rho(x\cos\theta+y\sin\theta)-3\rho^2,$$

$$\frac{x'}{x-2\rho\cos\theta}=\frac{y'}{y-2\rho\sin\theta}=R/\sqrt{\left[x^2+y^2-4\rho(x\cos\theta+y\sin\theta)+4\rho^2\right]},$$

where (x, y') is a point on $x^2+y^2=R^2$.

From these equations, we get

$$2\rho(x\cos\theta+y\sin\theta)-3r^2=R\sqrt{\left[x^2+y^2+\ldots\right]} \text{......(xxiii.)},$$

whence there results

$$x^2\left[R^2-4\rho^2\cos^2\theta\right]+y^2\left[R^2-4\rho^2\sin^2\theta\right]-8\rho^2\sin\theta\cos\theta xy+\ldots=0,$$

which is an ellipse.

Proceeding in the usual manner to find the centre, we obtain the equation $$h\sin\theta-k\cos\theta=0,$$

which shows that the centre is on OH.

If we put $y=x\tan\theta$ in (xxiii.), we obtain

$$2\rho x\sec\theta-3\rho^2=\pm R(x\sec\theta-2\rho),$$

or $$x_1\sec\theta(R-2\rho)=2\rho R-3\rho^2=l_1(R-2\rho),$$

$$x_2\sec\theta(R+2\rho)=2\rho R+3\rho^2=l_2(R+2\rho),$$

whence $$l_1+l_2=4\rho(R^2-3\rho^2)/(R^2-4\rho^2),$$

and $$l_1-l_2=2R\rho^2/(R^2-4\rho^2).$$

If we now turn the axes through an angle θ and move the origin to

the centre of the ellipse, we get its equation to be

$$x^2 \Big/ \frac{R^2\rho^4}{(R^2-4\rho^2)^2} + y^2 \Big/ \frac{\rho^4}{R^2-4\rho^2} = 1 \quad \text{.........(xxiv.)}.$$

From this equation it follows that the eccentricity is given by

$$e = 2\rho/R = 2OH/3R.$$

We may note here that $\angle OAH = C-B$,

therefore $$3\rho\cos\theta = R\sin(C-B),$$

$$3\rho\sin\theta = R\left[\cos(C-B)-2\cos A\right],$$

and (Booth, "New Geometrical Methods," Vol. II., p. 302)

$$9\rho^2 = 9R^2-(a^2+b^2+c^2)$$
$$= 9R^2-K \quad \text{.........................(xxv.)}.$$

If t_a, t_b, t_c are the tangents from the angular points, we have, from (x.),
$$t_c^2 = 2ab\cos C/3,$$

therefore $$3\Sigma t^2 = K.$$

Assume $2A_1$, $2B_1$, $2C_1$ to be the angles under which (O') is seen at the angular points, then $$\cot A_1 = t_a/\rho,$$

therefore $$\Sigma\cot^2 A_1 = \Sigma t^2/\rho^2 = K/3\rho^2,$$

and $$\Sigma\operatorname{cosec}^2 A_1 = (K+9\rho^2)/3\rho^2 = 3R^2/\rho^2.$$

The envelope of the polar of any point on (O'), with regard to circumcircle, may be found from the equations

$$xx'+yy' = R^2,$$

$$\frac{y'-2\rho\sin\theta}{y} = \frac{x'-2\rho\cos\theta}{x},$$

whence (xxii.), $$\frac{x^2+y^2}{\rho^2} = \frac{(x^2+y^2)^2}{\{R^2-2\rho(x\cos\theta+y\sin\theta)\}^2}.$$

This, when referred to OH, as axis of x, can be written

$$\rho^2(X^2+Y^2) = R^4-4\rho R^2X+4\rho^2X^2,$$

or $$3\rho^2X^2-\rho^2Y^2-4\rho R^2X+R^4 = 0,$$

making $$Y=0, \quad X_1 = R^2/\rho, \quad X_2 = R^2/3\rho;$$
whence the equation to the hyperbola referred to its centre is
$$X^2 - \frac{Y^2}{3} = \frac{R^4}{9\rho^2}.$$

The year has been prolific in publications bearing on the modern geometry of the triangle and circle.

Dr. Casey has brought out a third edition of his valuable "Sequel to Euclid," which now embodies in its Appendix a vast amount of information and of interesting results.

Captain Brocard sends us—"Sur des nouvelles propriétés du triangle" (*Journal de Math. élémentaires*, 1883); "Propriétés d'un groupe de trois paraboles" (*Mém. de. l'Acad. de Montpellier*, 1886); he also sends a "Démonstration de la proposition de Steiner relative à l'enveloppe de la droite de Simson" (*Bulletin de la Société Math. de France*, 1873).

Prof. Neuberg's paper, "Sur le Quadrilatère Harmonique" (referred to, Vol. XVI., p. 321), has been published in *Mathesis*, and with it is a Note, "Sur le point de Tarry"; he has written, also, "Sur le point de Steiner" (*Jour. de Math. Spéc.*, 1886), and has furnished, in co-operation with M. E. Lemoine, "Notes sur la Géometrie du triangle," to *Mathesis* (April, 1886).

The last-named writer sends several pamphlets; among them are—"Propriétés relatives à deux points ω, ω' du plan d'un triangle ABC qui se déduisent d'un point k quelconque du plan comme les points de Brocard se déduisent du point de Lemoine" (*Assoc. Franç. pour l'Avancement des Sciences*—Grenoble, 1885); to this is appended "Renseignements historiques et bibliographiques," which trace the history of this branch from very faint foreshadowings in 1809 to 1885 (close); "Sur une généralisation des propriétés relatives au cercle de Brocard et au point de Lemoine" (*Nouv. Annales*, Mai, 1885); "Théorèmes divers sur les Antiparallèles des côtés d'un triangle" (*Math.*, 1884); "Note sur quelques points remarquables du plan du triangle ABC" (*Math.*, Janvier, 1886); "Propriétés diverses du cercle et de la droite de Brocard" (*Math.*, Mai, 1885); "Quelques propriétés des parallèles et des antiparallèles aux côtés d'un triangle" (*Bulletin de la Soc. Math. de France*, 1884); "Sur les points associés du plan d'un triangle" (*Assoc. Franç. pour l'Avancement des Sciences*—Blois, Sept., 1884); "Exercices divers de Mathématiques élémentaires" (*Jour. de Math. élémentaires*, Paris, 1885).

Oberlehrer A. Artzt contributes "Untersuchungen über ähnliche Dreiecke, die einem festen Dreieck umschrieben sind, nebst einer

Anwendung auf die Gerade der zwölf harmonischen Punktreihen und ihre beiden Gegenbilder, die Ellipse und den Kreis der zwölf harmonischen Punktsysteme" (Kreis Brocard's), as a "Fortsetzung des Programms" von 1884 (des Gymnasiums zu Recklinghausen, 1886).

Mr. W. S. M'Cay's paper, "On three Circles related to a Triangle," is in the *Transactions of the R. Irish Academy* (July, 1885); and Dr. Casey's, "On the Harmonic Hexagon of a Triangle," is printed in the *Proceedings* (January 26, 1886) of the same Academy.

M. Morel has written an "Étude sur le cercle de Brocard," in the *Jour. de Math. élémentaires* (1883).

To the *Jornal de Sciencias Mathematicas e Astronomicas*, M. Maurice d'Ocagne furnishes an "Étude de Géométrie segmentaire"; the same mathematician supplies a "Monographie de la Symédiane" to the *Jour. de Math. élémentaires* (1886).

M. E. Vigarié (not Figarié, as in Vol. XVI., p. 321) is contributing *résumés* and proofs of results arrived at in papers which are being published in the *Jour. de Math. élémentaires*, viz., "Note de Géométrie" (31 pp., 1886), and "Propriétés générales des cercles de Tucker" (8 pp., 1886).

Dr. Lieber has sent portions of his *Zeitschrift*, XVI., pp. 577–608; on pp. 586–590 are a few problems connected with the Brocard circle.

Dr. P. H. Schoute has sent us a copy of "Over een nauwer Verband tusschen Hoek en cirkel von Brocard" (from the *Verslagen en Mededeelingen der K. Akad. van Wetenschappen*, 1886).

"On some Geometrical proofs of Theorems connected with the Inscription of a Triangle of constant form in a given Triangle," by Mr. M. Jenkins (*Quar. Jour. of Pure and Applied Math.*, Vol. XXI., No. 81), contains results connected with this subject, but it is mainly concerned with supplying Geometrical proof of results given by Mr. H. M. Taylor (*Lond. Math. Soc. Proc.*, Vol. XV., pp. 122–139).

R. T.

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INDEX OF PAPERS, &c.

[*The mark * indicates that the Paper has either been printed* in extenso, *or an abstract of it given in the "Proceedings," or in the Appendix.*]

GENERAL INDEX.

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LONDON MATHEMATICAL SOCIETY.

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CASH ACCOUNT.

Cr.

				£.	s.	d.
Balance from 1883-84:—						
General Fund	£292	9	3			
De Morgan Medal Fund...	10	19	11			
				303	9	2
Interest on Capital—						
Lord Rayleigh's Fund ...	50	11	1			
Life Composition Fund ...	21	10	11			
Invested Surplus Fund ...	10	3	10			
				82	5	10
12 Entrance Fees				12	12	0
162 Subscriptions—						
3 for 1880–81	3	3	0			
9 for 1881–82	9	9	0			
11 for 1882–83	11	11	0			
62 for 1883–84	65	2	3			
71 for 1884–85	74	11	0			
6 for 1885–86	6	6	0			
				170	2	3
Sales of *Proceedings* and Extra Copies				91	1	10
3 Life Compositions				31	10	0
De Morgan Medal Fund—						
Interest on Capital				3	2	10
				£694	3	11

Dr.

				£.	s.	d.
Printing *Proceedings*, &c.				204	18	9
Purchase of Journals				7	9	6
Binding				0	7	6
Postage, Stationery, Circulars, Bank Charges, and Sundries				14	19	3
Rent				10	0	0
Coals and Gas				1	10	0
Attendance—						
British Association ...	£1	5	0			
Mr. Stewardson	5	5	0			
				6	10	0
Life Composition Fund—						
Purchase of £20. 18*s*. 6*d*. Three per Cent. Consols				21	0	0
Invested Surplus Fund—						
Purchase of £200. 5*s*. New Three per Cents....				200	0	0
Balance at Bank—						
General Fund	£202	16	2			
De Morgan Medal Fund	14	2	9			
Life Composition Fund...	10	10	0			
				227	8	11
				£694	3	11

Audited and found correct,

10*th December*, 1885. (Signed) A. B. BASSET.

ASSETS AND LIABILITIES.

General Fund.

Assets		£.	s.	d.	Liabilities	£.	s.	d.
Cash at Bank		202	16	2	6 Subscriptions for 1885-86 paid in advance	6	6	0
58 Subscriptions due and owing—								
1 for 1881–2	£1 1 0							
1 for 1882–3	1 1 0							
12 for 1883–4	12 12 0							
*44 for 1884–5	46 4 0							
		60	18	0	Balance	257	8	2
		£263	14	2		£263	14	2

* Viz., 117, the number of Subscribers for 1884-85, less 2 Subscriptions paid in advance in 1883-84, and 71 paid during 1884-85.

De Morgan Medal Fund.

Assets	£.	s.	d.	Liabilities	£.	s.	d.
Cash at Bank, the Proceeds of $4\frac{1}{2}$ years' Dividends on Invested Capital	14	2	9	Balance	14	2	9

CAPITAL ACCOUNT.

	Sum Invested. £.	s.	d.	£.	s.	d.	Description of Investment.
General Fund—							
1. Life Composition Fund	752	10	0	762	5	6	Three per Cent. Consols.
				10	10	0	Cash at Bank waiting for Investment.
2. Lord Rayleigh's Fund	1000	0	0	870	0	0	Guaranteed Five per Cent. Great Indian Peninsular Railway Stock.
3. Invested Surplus Fund	350	0	0	350	16	3	New Three per Cents.
De Morgan Medal Fund	103	5	3	104	19	8	Reduced Three per Cents.

Audited and found correct,

10*th December*, 1885. (Signed) A. B. BASSET.

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OF THE

LONDON MATHEMATICAL SOCIETY.

12TH NOVEMBER, 1885.

TWENTY-SECOND SESSION, 1885-6.

LONDON:
PRINTED BY C. F. HODGSON & SON,
GOUGH SQUARE, FLEET STREET.
1885.

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All Subscriptions should be paid to the Treasurer, A. B. KEMPE, Esq., 7 Crown Office Row, Temple, E.C., by Crossed Cheque, or by Money Order, payable at the Charing Cross Post Office.

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NOVEMBER 12, 1885.

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* indicates that the Member was a Member of the Society at its formation, January 16th, 1865.

† indicates that the Member has paid the life-composition, and is a Member for life.

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Date of Election.

†1866 May 21. ADAMS, WILLIAM GRYLLS, M.A., F.R.S., F.G.S., F.C.P.S.; late Fellow of St. John's College, Cambridge; Professor of Natural Philosophy and Astronomy in King's College, London; Permanent Vice-President of the Physical Society; 43 *Notting Hill Square, W.; and Athenæum Club, S.W.*

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1882 Jan. 12. ALLMAN, GEORGE JOHNSTON, LL.D. Dublin, D.Sc., F.R.S.; Professor of Mathematics in Queen's College, Galway; Senator of the Royal University of Ireland; *St. Mary's, Galway.*

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†1885 April 2. BALL, ROBERT STAWELL, LL.D., F.R.S.; Andrews Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland; *The Observatory, Dunsink, Co. Dublin.*

1880 April 8. BARNARD, JAMES, M.A., St. John's College, Cambridge; Head Mathematical Master at Christ's Hospital.

1885. Jan. 8. BARRELL, FRANCIS RICHARD, B.A. Cambridge; B.Sc. London; late Scholar of Pembroke College, Cambridge; Instructor in Natural Science, H.M.S. Britannia; *Dartmouth.*

†1883 Dec. 13. BASSET, ALFRED BARNARD, M.A., Trinity College, Cambridge; 11 *Old Square, Lincoln's Inn; and Chapel Place Mansion,* 322 *Oxford Street, W.*

1885. April 2. BASU, BABU SYAMA CHARAN, B.A., Calcutta University; Senior Mathematical Lecturer at the South Suburban School; 2 *G. P. Basu's Lane, Chaulputty, Bhowanipore, Calcutta.*

Date of Election.

1881 Feb. 10. BEMAN, WOOSTER WOODRUFF, A.B., A.M., University of Michigan; Assistant Professor of Mathematics in the University of Michigan; *Ann Arbor, Michigan, U.S.A.*

1882 June 8. BERRY, JOHN W., *Pittston, Luzerne County, Pennsylvania, U.S.A.*

†1866 Dec. 13. BESANT, WILLIAM HENRY, M.A., D.Sc., F.R.S., F.R.A.S., F.C.P.S.; Mathematical Lecturer at, and late Fellow of, St. John's College, Cambridge; *Spring Lawn, Harvey Road, Cambridge.*

†1875 Feb. 11. BICKMORE, CHARLES EDWARD, M.A.; Fellow of New College, Oxford; *Oxford.*

1884 Mar. 13. BLOMFIELD, Rev. ALFRED CHARLES EDWARD, M.A. Oxon.; *Campsall Vicarage, Doncaster, Yorkshire.*

* BRIDGE, JOHN, M.A. London; 56 *South Hill Park, Hampstead Heath, N.W.*

1884 May 8. BRILL, JOHN, M.A., Fellow of St. John's College, Cambridge; *Cambridge.*

1884 Jan. 10. BROCKELBANK, DUNCAN, late Assistant Actuary, Continental Life Insurance Company, New York; *Post Office, Adelaide, South Australia.*

†1866 Feb. 19. BROOK-SMITH, JOHN, M.A.; St. John's College, Cambridge; Mathematical Master in Cheltenham College; Barrister-at-Law; *Boyne House, Cheltenham.*

1882 Jan. 12. BRYANT, SOPHIA, D.Sc. London, F.C.P.; North London Collegiate School for Girls, Sandall Road; 2 *Anson Road, Tufnell Park, N.*

1882 Mar. 9. BUCHHEIM, ARTHUR, M.A.; late Scholar of New College, Oxford; *The Grammar School, Manchester.*

†1879 Dec. 11. BURNSIDE, WILLIAM, M.A.; Fellow of Pembroke College, Cambridge; Professor of Mathematics, Royal Naval College; *Greenwich.*

1880 Jan. 8. BURNSIDE, WILLIAM SNOW, M.A.; Erasmus Smith's Professor of Mathematics, Trinity College, Dublin; *Dublin.*

†1875 Dec. 9. CAMPBELL, JOHN ROBERT, F.G.S., Lieut.-Colonel, late Hants Artillery Militia; *Charing, Ashford, Kent;* and *Union Club, Trafalgar Square, W.C.*

†1874 Mar. 4. CARPMAEL, ERNEST, M.A., F.R.A.S.; late Fellow of St. John's College, Cambridge; Barrister-at-Law; 3 *Harcourt Buildings, Temple, E.C.*

†1873 Nov. 13. CARR, GEORGE SHOOBRIDGE, M.A.; late Scholar of Gonville and Caius College, Cambridge; 3 *Endsleigh Gardens, N.W.*

†1874 Nov. 12. CASEY, JOHN, LL.D., F.R.S., V.-P.R.I.A.; Professor of Higher Mathematics and Mathematical Physics in the Catholic University, Ireland; Fellow of the Royal University of Ireland; Member of the Paris Mathematical Society; 86 *South Circular Road, Dublin.*

1865 June 19. CAYLEY, ARTHUR, M.A., F.R.S.L. & E., LL.D. Edinburgh, V.-P.R.A.S., Fellow of Trinity College, Cambridge; Corresponding Member of the Institute of France; Sadlerian Professor of Mathematics in the University of Cambridge; Dr. of Mathematics and Physics, Leyden; Copley Medallist of the Royal Society, 1883; De Morgan Medallist, 1884; Foreign Associate of the United States National Academy of Sciences; *Garden House, Cambridge.*

Date of Election.

†1884 Mar. 13. CHEVALLIER, JOHN, B.A. Cambridge and Oxford; late of Trinity College, Cambridge; Fellow of New College, and Mathematical Tutor of Magdalen College, Oxford; *Oxford.*

* CLIFTON, ROBERT BELLAMY, M.A. Cambridge and Oxford, F.R.S., F.R.A.S., F.C.P.S.; Fellow of Wadham and Merton Colleges, Oxford; late Fellow of St. John's College, Cambridge; Professor of Experimental Philosophy in the University of Oxford; Vice-President of the Physical Society; *Portland Lodge, Park Town, Oxford;* and *Athenæum Club, London, S.W.*

†1871 April 13. CLOSE, FREDERICK, Colonel, Royal Artillery; *Woolwich.*

†1870 June 9. COCKLE, SIR JAMES, Knt., M.A. Trinity College, Cambridge; F.R.S., F.R.A.S., F.C.P.S.; Corresponding Member of the Literary and Philosophic Society of Manchester; Honorary Member of the Royal Society of New South Wales; late President of the Queensland Philosophical Society; and Chief Justice of Queensland, Australia; 12 *St. Stephen's Road, Westbourne Park, W.*

†1876 Mar. 16. COCKSHOTT, ARTHUR, M.A.; late Fellow and Assistant Tutor of Trinity College, Cambridge; *Eton College, Windsor.*

1867 Jan. 24. CORRIE, JOSIAH OWEN, B.A. Cambridge; F.R.A.S.; Barrister-at-law; *Casanova, Upper Eglinton Road, Shooter's Hill, Woolwich.*

1865 Oct. 16. COTTERILL, JAMES HENRY, M.A., F.R.S., Member of Council of the Physical Society; late Scholar of St. John's College, Cambridge; Professor of Applied Mathematics, Royal Naval College, Greenwich; 18 *Gloucester Place, Greenwich, S.E.*

1881 Jan. 13. CRAIG, THOMAS, Ph.D.; United States Coast Survey; Associate Professor of Applied Mathematics at Johns Hopkins University; Member of the Paris Mathematical Society; *Baltimore, U.S.A.*

†1882 Dec. 14. CUNNINGHAM, ALLAN JOSEPH CHAMPNEYS, Major R.E.; Associate, Institute of Civil Engineers; Fellow of King's College, London; *Brompton Barracks, Chatham.*

1882 May 11. DANIELS, ARCHIBALD LAMONT, A.B.; Fellow of Johns Hopkins University; *Baltimore, U.S.A.*

†1868 Nov. 26. DARWIN, GEORGE HOWARD, M.A., F.R.S., F.R.A.S., F.C.P.S.; LL.D. Edinburgh; Fellow of Trinity College, and Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge; *Newnham Grange, Cambridge.*

1876 Dec. 14. DAVIS, ROBERT FREDERIC, M.A.; late Scholar of Queens' College, Cambridge; 70 *Torrington Square, Bloomsbury, W.C.*

1881 June 9. DICK, GEORGE ROGER, M.A.; late Fellow of Gonville and Caius College, Cambridge; Professor of Mathematics in the Royal University, Mauritius; *Mauritius.*

†1878 Nov. 14. DICKSON, JAMES DOUGLAS HAMILTON, M.A., F.R.S.E., F.C.P.S.; Fellow and Tutor of St. Peter's College, Cambridge; *Cambridge.*

Date of Election.

1885. Jan. 8. DUTT, MAHENDRA NATH, B.A., Calcutta University; Senior Professor of Mathematics, St. Stephen's College, Delhi; Mahárajá of Vizianagram Medallist; *St. Stephen's College, Delhi.*

1880 Feb. 12. EDWARDES, DAVID; 7 *Erith Villas, South Road, Erith, Kent.*

†1875 Jan. 14. ELLIOTT, EDWIN BAILEY, M.A.; Fellow of, and Lecturer at, Queen's College, Oxford; *Oxford.*

†1865 June 19. ELLIS, ALEXANDER JOHN, B.A. Cambridge; F.R.S., F.C.P.S., F.S.A.; Vice-President of the Philological Society; 25 *Argyll Road, Kensington, W.*

1884 June 12. ELY, GEORGE STETSON, Ph.D.; late Fellow of Johns Hopkins University; *Patent Office, Room* 98, *Washington, D.C., U.S.A.*

1866 June 18. ESSON, WILLIAM, M.A., F.R.S., F.R.A.S.; Member of the Physical Society; Fellow and Tutor of Merton College; Lecturer at Magdalen and Corpus Christi Colleges, Oxford; 1 *Bradmore Road, Oxford.*

1866 June 26. FAULKNER, CHARLES JOSEPH, M.A.; Member of the Physical Society; Fellow and Tutor of University College, Oxford; *Oxford.*

1882 June 8. FORSYTH, ANDREW RUSSELL, M.A., F.C.P.S.; Member of the Physical Society; Fellow and Assistant Tutor of Trinity College, Cambridge; *Trinity College, Cambridge.*

†1883 Dec. 13. FORTEY, HENRY, M.A.; late Scholar of Gonville and Caius College, and of Jesus College, Cambridge; late Acting Principal and Professor of Mathematics, Presidency College, Madras; Inspector of Schools, Madras Presidency, and Fellow of the Madras University; 18 *Hughenden Road, Clifton.*

†1865 June 19. FOSTER, JOHN EBENEZER, M.A. Trinity College, Cambridge; 2 *Scroope Terrace, Cambridge.*

†1866 Mar. 19. FOWLER, Sir ROBERT NICHOLAS, Bart., M.A. London; F.R.G.S., M.P.; Alderman of the City of London; Fellow of University College, London; 50 *Cornhill, E.C.*

1881 May 12. FRANKLIN, FABIAN, Mrs., Johns Hopkins University; *Baltimore, Maryland, U.S.A.*

1881 May 12. FRANKLIN, FABIAN, Ph.D.; Johns Hopkins University; *Baltimore, Maryland, U.S.A.*

1882 Dec. 14. FRASER, HUGH, M.A., late Scholar of Trinity Hall, Cambridge; *Woodhouse, Harrow-on-the-Hill.*

†1871 Dec. 14. FREEMAN, Rev. ALEXANDER, M.A., F.R.A.S., F.C.P.S.; Member of the Physical Society; late Fellow of St. John's College, Cambridge; *Murston Rectory, Sittingbourne, Kent.*

1869 Dec. 9. FROST, Rev. PERCIVAL, M.A., D.Sc., F.C.P.S., F.R.S.; Fellow of, and Mathematical Lecturer at, King's College; and late Fellow of St. John's College, Cambridge; 15 *Fitzwilliam Street, Cambridge.*

1872 June 13. GENESE, ROBERT WILLIAM, M.A.; late Scholar of St. John's College, Cambridge; Professor of Mathematics, University College of North Wales; *Aberystwith.*

Date of Election.

†1883 Jan. 11. GERRANS, HENRY TRESAWNA, M.A.; Fellow of, and Mathematical Lecturer at, Worcester College, Oxford; *Oxford.*

†1868 Feb. 27. GLAISHER, JAMES, F.R.S., F.R.A.S.; 1 *Dartmouth Place, Blackheath.*

†1872 Feb. 8. GLAISHER, JAMES WHITBREAD LEE, M.A., F.R.S., F.R.A.S., F.C.P.S.; Fellow and Tutor of Trinity College, Cambridge; PRESIDENT; *Cambridge.*

1883 Dec. 13. GLAZEBROOK, RICHARD TETLEY, M.A., F.R.S., Fellow and Assistant-Tutor of Trinity College, and Demonstrator of Physics in the University of Cambridge; Member of Council of the Physical Society; Secretary of the Cambridge Philosophical Society; *Cambridge.*

1873 Mar. 13. GREENHILL, ALFRED GEORGE, M.A., F.C.P.S.; late Fellow of Emmanuel College, Cambridge; Examiner in Mathematics and Natural Philosophy in the University of London; Professor of Mathematics to the Advanced Class of Officers, Woolwich; *Royal Artillery Institution, Woolwich.*

1871 June 8. GRIFFITHS, JOHN, M.A.; Fellow and Tutor of Jesus College, Oxford; *Oxford.*

1874 May 14. HAMMOND, JAMES, M.A.; late Scholar of Queen's College, Cambridge; *Buckhurst Hill, Essex.*

†1865 Mar. 20. HARDING, PERCY JOHN, M.A. Cambridge; *Bedford College,* 8 *York Place, Portman Square.*

1879 Mar. 13. HARGREAVES, RICHARD, M.A.; Fellow of St. John's College, Cambridge; Mathematical Master at Merchant Taylors' School; 36 *Lincoln's Inn Fields, W.C.*

†1865 Oct. 16. HARLEY, Rev. ROBERT, F.R.S., F.R.A.S.; Corresponding Member of the Literary and Philosophical Society of Manchester; Honorary and Corresponding Member of the Queensland Philosophical Society; late Principal of The College, Huddersfield; 96 *Netherwood Road, W.*

1874 Dec. 10. HART, HARRY, M.A.; late Fellow of Trinity College, Cambridge; Professor of Mathematics and Applied Mechanics in the Royal Military Academy, Woolwich; *Cromer House, Lee Terrace, Blackheath.*

1865 Oct. 16. HAYDON, FRANK SCOTT, B.A. Cambridge; *Public Record Office, Rolls House, Chancery Lane, W.C.;* and *Southey Lodge, Kingston Road, South Wimbledon, S.W.*

†1884 Mar. 13. HAYES, EDWARD HAROLD, M.A., Fellow of, and Mathematical Lecturer at, New College, Oxford; *Oxford.*

1871 Feb. 9. HAYWARD, ROBERT BALDWIN, M.A., F.R.S., F.C.P.S.; President of the Association for the Improvement of Geometrical Teaching; Mathematical Master at Harrow School; Fellow of University College, London; late Fellow of St. John's College, Cambridge; *The Park, Harrow-on-the-Hill.*

1884 Mar. 13. HEATH, ROBERT SAMUEL, M.A. Cambridge; D.Sc. London; Fellow of Trinity College, Cambridge; Professor of Mathematics in the Mason Science College, Birmingham; *Birmingham.*

Date of Election.

1868 April 23. HENRICI, OLAUS M. F. E.; Dr. Phil. Heidelberg; LL.D. St. Andrews; F.R.S.; Member of the Physical Society; Professor of Mechanics and Mathematics in the City and Guilds of London Institute, Central Institution, *Exhibition Road, S.W.*; VICE-PRESIDENT; *Meldorf Cottage, Kemplay Road, Hampstead, N.W.*

1883 Dec. 13. HEPPEL, GEORGE, M.A., St. John's College, Cambridge; 180 *The Grove, Hammersmith.*

1878 May 9. HICKS, WILLIAM MITCHINSON, M.A., F.R.S., F.C.P.S.; Member of the Physical Society; Fellow of St. John's College, Cambridge; Principal of Firth College; *Endcliffe Crescent, Sheffield.*

†1883 May 10. HILL, MICAIAH JOHN MULLER, M.A., Fellow of St. Peter's College, Cambridge; Examiner in Mathematics and Natural Philosophy in the University of London; Professor of Mathematics in University College, London; *Gower Street, W.C., and* 71 *Southboro Road, South Hackney, E.*

† * HIRST, THOMAS ARCHER, Ph.D. Marburg, F.R.S., F.R.A.S., F.C.P.S.; Member of Council of University College, London; Member of the Physical Society; Member of the Paris Mathematical Society; 7 *Oxford and Cambridge Mansions, Marylebone Road, N.W.*, and *Athenæum Club, S.W.*

1871 Mar. 9. HODGSON, CHARLES ROBERT, B.A. London; Secretary of the College of Preceptors; 29 *Milner Square, Islington, N.*

†1873 Feb. 13. HOPKINSON, JOHN, M.A. Cambridge, D.Sc. London, F.R.S.; Vice-President of the Physical Society; late Fellow of Trinity College, Cambridge; 3 *Holland Villas Road, Kensington, W.*

1868 May 28. HUDSON, WILLIAM HENRY HOAR, M.A., LL.M., F.C.P.S.; late Fellow of St. John's College, Cambridge; Professor of Mathematics, King's College, London, and Queen's College, Harley Street; External Examiner in Mathematics to the Victoria University; 14 *Geraldine Road, Wandsworth, S.W.*

1884. Dec. 11. IBBETSON, WILLIAM JOHN, B.A., F.R.A.S., F.C.P.S.; late Senior Scholar of Clare College, Cambridge; 26 *Bateman Street, Cambridge.*

†1879 Dec. 11. JACK, WILLIAM, M.A., LL.D.; Member of the Physical Society; Professor of Mathematics in the University of Glasgow; late Fellow of St. Peter's College, Cambridge; *Glasgow.*

1875 Jan. 14. JEFFERY, HENRY MARTYN, M.A. Cambridge, F.R.S.; late Head Master of the Grammar School, Cheltenham; 9, *Dunstanville Terrace, Falmouth, Cornwall.*

†1865 May 15. JENKINS, MORGAN, M.A.; late Scholar of Christ's College, Cambridge; **Honorary Secretary**; 50 *Cornwall Road, Westbourne Park, W.*

1879 Dec. 11. JOHNSON, WILLIAM WOOLSEY, M.A.; Professor of Mathematics in the United States Naval Academy; *Annapolis, Maryland, U.S.A.*

1875 Mar. 11. KEMPE, ALFRED BRAY, M.A., F.R.S.; Barrister-at-Law; late Scholar of Trinity College, Cambridge; **Treasurer**; 7 *Crown Office Row, Temple, E.C.*

Date of Election.

1885. Mar. 12. LACHLAN, ROBERT, B.A.; Fellow of Trinity College, Cambridge; *Cambridge.*

†1873 Dec. 11. LAMB, HORACE, M.A., F.R.S.; late Fellow of Trinity College, Cambridge; Professor of Mathematics in the Owens College, Victoria University; *Manchester.*

†1874 Jan. 8. LAMBERT, CARLTON JOHN, M.A. Cambridge, F.R.A.S.; Member of the Physical Society; late Fellow of Pembroke College, Cambridge; Professor of Mathematics, Royal Naval College; *Greenwich, S.E.*

1884 Mar. 13. LARMOR, JOSEPH, M.A., D.Sc. London; Fellow of, and Lecturer at, St. John's College, Cambridge; Fellow of the Royal University of Ireland; late Professor of Natural Philosophy in Queen's College, Galway; *Cambridge.*

1874 Feb. 12. LAVERTY, Rev. WALLIS HAY, M.A.; late Fellow of Queen's College, Oxford; Rector of Headley; *Liphook, Hants.*

†1875 Jan. 14. LEUDESDORF, CHARLES, M.A.; Member of the Physical Society; Fellow of, and Lecturer at, Pembroke College, Oxford; *Oxford.*

†1883 Mar. 8. LODGE, ALFRED, M.A.; Fereday Fellow of St. John's College, Oxford; Assistant Professor of Mathematics in the Royal Indian Engineering College; *Coopers Hill, Staines.*

1873 Jan. 9. MACLEOD, JOHN, University of Glasgow; one of Her Majesty's Inspectors of Schools, Scotland; *Elgin.*

1882 Dec. 14. MACMAHON, PERCY ALEXANDER, Captain R.A.; Instructor of Mathematics, Royal Military Academy, Woolwich; *Woolwich.*

1885. Mar. 12. MAGNUS, PHILIP, B.A., B.Sc. London; Director and Secretary of the City and Guilds of London Technical Institute, Gresham College, E.C.; 48 *Gloucester Place, Portman Square.*

1880 Mar. 11. MCCLINTOCK, EMORY, Actuary of the North-Western Life Insurance Company; *Milwaukee, Wisconsin, U.S.A.*

1868 May 28. MAIN, PHILIP THOMAS, M.A., F.C.P.S.; Fellow of, and Lecturer in Natural Science at, St. John's College, Cambridge; *Cambridge.*

1878 April 11. MARTIN, ARTEMAS, M.A.; Member of the Paris and Edinburgh Mathematical Societies; 117 *East Tenth Street, Erie, Pennsylvania, U.S.A.*

†1883 Nov. 8. MILNE, Rev. JOHN JAMES, M.A.; late Scholar of St. John's College, Cambridge; Second Master of Heversham Grammar School; *Chestnut House, Heversham, Milnthorpe, Westmoreland.*

1875 Dec. 9. MINCHIN, GEORGE MINCHIN, M.A.; Trinity College, Dublin; Member of the Physical Society; Professor of Applied Mathematics at the Royal Indian Engineering College; *Coopers Hill, Staines.*

1881 Jan. 13. MITCHELL, OSCAR HOWARD, Fellow of Johns Hopkins University; *Marietta College, Marietta, Ohio, U.S.A.*

1883 Jan. 11. MOLLISON, WILLIAM LOUDON, M.A., Fellow of Clare College, Cambridge; *Cambridge.*

Date of Election.

1868 Mar. 26. MOULTON, JOHN FLETCHER, M.A., F.R.S., F.R.A.S., F.C.P.S., Q.C.; Member of the Physical Society; late Fellow of Christ's College, Cambridge; Barrister-at-Law; Vice-President of the Association for the Improvement of Geometrical Teaching; 74 *Onslow Gardens, S.W.*

1874 April 9. MUIR, THOMAS, M.A. Glasgow, LL.D. Edinburgh, F.R.S.E.; Mathematical Master in the High School of Glasgow; late President of the Edinburgh Mathematical Society; *Beechcroft, Bishopton, N.B.*

†1884 Jan. 10. MUKHOPÂDHYÂY, ASUTOSH, B.A., F.R.A.S.; Scholar of Presidency College, and of the University of Calcutta; 77 *Russapaglah Road, Bhowanipore, Calcutta.*

†1874 Dec. 10. NANSON, EDWARD JOHN, M.A., late Fellow of Trinity College, Cambridge; Professor of Mathematics in the University of Melbourne; *Melbourne.*

1874 April 9. NIVEN, CHARLES, M.A., D.Sc., F.R.S., F.R.A.S.; late Fellow of Trinity College, Cambridge; Professor of Natural Philosophy in the University of Aberdeen; 6 *Chancery, Old Aberdeen.*

1873 May 8. NIVEN, WILLIAM DAVIDSON, M.A., F.R.S., F.R.A.S., F.C.P.S.; Fellow of Trinity College, Cambridge; Member of the Physical Society; *Royal Naval College, Greenwich, S.E.*

1865 Dec. 18. NOBLE, ANDREW, Captain (late R.A.), F.R.S., F.R.A.S.; *Jesmond Dene, Newcastle-on-Tyne;* and *Athenæum Club, S.W.*

1872 April 11. PAICE, WILLIAM, M.A. London; University College School; 28 *Caversham Road, N.W.*

1884 Nov. 13. PEARSON, KARL, M.A.; Fellow of King's College, Cambridge; Professor of Applied Mathematics in University College, London; 2 *Harcourt Buildings, Temple, E.C.*

1880 Mar. 11. PEIRCE, CHARLES SANDERS, A.M. and S.B. Harvard; of the United States Coast and Geodetic Survey; Lecturer on Logic at the Johns Hopkins University; Member of the National Academy of Sciences; *United States Coast and Geodetic Survey, Washington D. C., U.S.A.*

1877 April 12. PENDLEBURY, CHARLES, M.A., F.R.A.S.; late Scholar of St. John's College, Cambridge; *St. Paul's School, Hammersmith, W.*; and 4 *Glazbury Road, West Kensington, W.*

1868 Jan. 23. PERIGAL, HENRY, F.R.A.S.; Member of the Physical Society; 9 *North Crescent, Bedford Square, W.C.*

1885. Feb. 12. PERRIN, EMILY; late of Girton College; *Ladies' College, Cheltenham.*

1878 Jan. 10. PHILLIPS, FRANCIS BARCLAY WILMER, M.A. and M.B. Oxford; B.Sc. London; M.R.C.S.; 75 *Lansdowne Place, Brighton.*

1881 Nov. 10. PIRIE, Rev. GEORGE, M.A., Fellow of Queen's College, Cambridge; Professor of Mathematics in the University of Aberdeen; *Aberdeen.*

1868 June 25. DE POLIGNAC, Prince CAMILLE; Member of the Paris Mathematical Society; 24 *Charles Street, Berkeley Square, W.*

Date of Election.

†1866 June 26. PRICE, Rev. BARTHOLOMEW, M.A., F.R.S., F.R.A.S., Hon. F.C.P.S.; Member of the Physical Society; Member of the Paris Mathematical Society; Fellow and Vice-Gerent of Pembroke College, Oxford; Sedleian Professor of Natural Philosophy in the University of Oxford; Curator of the Bodleian Library; 11 *St. Giles's, Oxford.*

1866 June 26. PRICE, CHARLES JAMES COVERLEY, M.A.; Fellow of, and Lecturer at, Exeter College, Oxford; *Oxford.*

1865 Oct. 16. RACSTER, WILLIAM; 17 *Cambridge Terrace, Crystal Palace Road, East Dulwich.*

† * RANYARD, ARTHUR COWPER, M.A., F.R.A.S.; Pembroke College, Cambridge; Member of the Physical Society; Barrister-at-Law; 25 *Old Square, Lincoln's Inn, W.C.*

1885. May 14. RAU, B. HANUMANTA, B.A.; Acting Headmaster, Government Normal School; *Madras.*

†1873 June 12. RAWSON, ROBERT, F.R.A.S.; *Havant, Hants.*

†1871 April 13. RAYLEIGH, JOHN WILLIAM, LORD, M.A. Cambridge; D.C.L. Oxon.; LL.D. Edinburgh; F.R.S., F.R.A.S., F.R.G.S.; Honorary Fellow of Trinity College, Cambridge; Vice-President of the Cambridge Philosophical Society; Vice-President of the Physical Society; *Terling Place, Witham, Essex.*

1875 June 10. RHODES, EDWARD HAWKSLEY, B.A. Cambridge; Deputy Keeper in Her Majesty's Office of Land Revenue Records; Vice-President of the Aristotelian Society; 11 *Norfolk Road, St. John's Wood, N.W.*

1874 June 11. RITCHIE, WILLIAM IRVINE, B.A.; late Scholar of Trinity College, Cambridge; *Education Department, Whitehall;* and *Southmead, Wimbledon Park.*

1880 Dec. 9. ROBERTS, RALPH AUGUSTUS, M.A.; Scholar and Senior Mathematical Moderator of Trinity College, Dublin; 23 *Clyde Road, Dublin.*

1865 June 19. ROBERTS, SAMUEL, M.A. London, F.R.S.; 126 *Tufnell Park Road, Holloway, N.*

†1885. Jan. 8. ROBERTS, SAMUEL OLIVER, B.A.; Scholar of St. John's College, Cambridge; Senior Mathematical Master at the Royal Grammar School, Newcastle-on-Tyne; 4 *Gloucester Terrace, Newcastle-on-Tyne.*

1880 Dec. 9. ROBERTS, WILLIAM RALPH WESTROPP, M.A.; Fellow of Trinity College, Dublin; 2 *Wellington Road, Dublin.*

1885. Nov. 12. ROGERS, LEONARD JAMES, B.A., B.Mus., Balliol College; Mathematical Lecturer at Wadham College, Oxford; *Oxford.*

1881 May 12. ROSENTHAL, LIONEL, B.A. Dublin; 4 *New Square, Lincoln's Inn, W.C.*

* ROUTH, EDWARD JOHN, M.A., D.Sc. Cambridge; LL.D. Glasgow; F.R.S., F.R.A.S., F.G.S., F.C.P.S.; Honorary Fellow and late Assistant Tutor of St. Peter's College, Cambridge; *Newnham Cottage, Cambridge.*

†1875 Jan. 14. RUSSELL, JOHN WESLEY, M.A.; Member of the Physical Society; Fellow of Merton College, and Lecturer at Balliol College, Oxford; *Oxford.*

Date of Election.

†1866 April 16. SALMON, Rev. GEORGE, D.D. Dublin and Edinburgh; D.C.L., LL.D., F.R.S., Hon. F.C.P.S.; Regius Professor of Divinity at, and Fellow of, Trinity College, Dublin; Corresponding Member of the Institute of France; *Dublin.*

1865 June 19. SAVAGE, THOMAS, M.A., F.C.P.S.; late Fellow of Pembroke College, Cambridge; Extra-Assistant Examiner in Mathematics and Natural Philosophy in the University of London; 2 *St. Bartholomew Road, Tufnell Park, N.*, and *New University Club, St. James's Street, S.W.*

1881 Jan. 13. SCOTT, CHARLOTTE ANGAS, D.Sc. London; *Lancashire College, Whalley Range, Manchester*; and *Bryn Mawr College, Philadelphia, U.S.A.*

1874 Jan. 8. SCOTT, ROBERT FORSYTH, M.A.; Fellow and Senior Bursar of St. John's College, Cambridge; Barrister-at-Law; Member of the Edinburgh Mathematical Society; *Cambridge.*

†1879 Dec. 11. SHARP, WILLIAM JOSEPH CURRAN, M.A.; late Scholar of Trinity College, Dublin; 14 *Mount Street, Grosvenor Square, W.*

1884. Dec. 11. SIMMONS, Rev. THOMAS CHARLES, M.A., Magdalene College, Cambridge; Mathematical Master of Christ's College, Brecon; 14 *Bridge Street, Brecon.*

†1875 Jan. 14. SMITH, CHARLES, M.A.; Fellow and Tutor of Sidney-Sussex College, Cambridge; *Cambridge.*

1870 Dec. 8. SMITH, JAMES HAMBLIN, M.A., Gonville and Caius College; late Lecturer at St. Peter's College, Cambridge; 42 *Trumpington Street, Cambridge.*

1881 Jan. 13. SMITH, JAMES PARKER, M.A.; Fellow of Trinity College, Cambridge; Barrister-at-Law; 3 *Stone Buildings, Lincoln's Inn, W.C.*

†1868 Nov. 26. SPRAGUE, THOMAS BOND, M.A., F.R.S.E., F.C.P.S.; late Fellow of St. John's College, Cambridge; Scottish Equitable Life Assurance Office; 26 *St. Andrew Square, Edinburgh.*

1874 June 11. STEVENS, Rev. ALFRED JAMES, M.A.; Fellow of St. John's College, Cambridge; 15 *David Place, Jersey.*

1866 Feb. 19. STIRLING, JAMES, M.A.; Trinity College, Cambridge; Member of the Physical Society; Barrister-at-Law; *Union Bank Chambers*, 61 *Carey Street, W.C.*; and 73 *Ladbroke Grove, Notting Hill, W.*

†1879 Mar. 13. STORY, WILLIAM E., A.B. Harvard; Ph.D. Leipzic; Associate-Professor of Mathematics at Johns Hopkins University; *Baltimore, U.S.A.*

1881 Dec. 8. STUART, GEORGE HENRY, M.A.; late Fellow of Emmanuel College, Cambridge; *Kumbakonam, Madras.*

1865 June 19. SYLVESTER, JAMES JOSEPH, M.A. Oxford and Cambridge; D.C.L. Oxon.; LL.D. Dublin and Edinburgh; F.R.S. L. and E., Honorary Fellow of St. John's College, Cambridge, and Fellow of New College, Oxford; Savilian Professor of Geometry in the University of Oxford; Foreign Associate of the United States National Academy of Sciences; Foreign Member of the Royal Academy of Sciences, Göttingen; VICE-PRESIDENT; *New College, Oxford.*

Date of Election.

1875 Nov. 11. Tanner, Henry William Lloyd, M.A. Oxford; Associate of the Royal School of Mines; Professor of Mathematics and Astronomy in the University College of South Wales and Monmouthshire; *The Abbotts, Clive Road, Penarth, Cardiff.*

†1872 June 13. Taylor, Rev. Charles, M.A., D.D., F.C.P.S., F.S.A.; Master of St. John's College, Cambridge; *Cambridge.*

†1866 Mar. 19. Taylor, Henry Martyn, M.A., F.R.A.S., F.C.P.S.; Fellow and late Tutor of Trinity College, Cambridge; *Cambridge.*

†1880 Mar. 11. Temperley, Ernest, M.A.; Fellow of, and Mathematical Lecturer at, Queens' College, Cambridge; *Cambridge.*

†1878 May 9. Terry, Rev. Thomas Robert, M.A. Cambridge and Oxford; F.R.A.S.; formerly Scholar of Trinity College, Cambridge; late Fellow and Tutor of Magdalen College, Oxford; *The Rectory, East Ilsley, Berks.*

1866 Dec. 13. Thomas, Rev. David, M.A.; late Fellow of Trinity College, Oxford; *Garsington Rectory, Oxford.*

†1883 Dec. 13. Thomson, Joseph John, M.A., F.R.S., F.C.P.S.; Professor of Experimental Physics in the University of Cambridge; Fellow of Trinity College, Cambridge; *Cambridge.*

†1867 Nov. 28. Thomson, Sir William, Knt., M.A. Cambridge; LL.D., D.C.L., F.R.S. L. and E., F.R.A.S., F.C.P.S.; Vice-President of the Physical Society; Foreign Associate of the United States National Academy of Sciences; Hon. Member of the New Zealand Institute; Fellow of St. Peter's College, Cambridge; Professor of Natural Philosophy in the University of Glasgow; Foreign Associate of the Academy of Sciences, Paris, and of the Royal Society of Göttingen; *Glasgow.*

1883 Dec. 13. Thomson, William, M.A., B.Sc. Edinb.; F.R.S.E.; Professor of Mathematics in Stellenbosch College; *Cape Colony.*

1882 Feb. 9. Tompson, J. H., F.C.S., F.G.S.; Science Master, Church of England Grammar School; *Melbourne.*

1865 Oct. 16. TUCKER, ROBERT, M.A.; late Scholar of St. John's College, Cambridge; Mathematical Master in University College School; Vice-President of the Association for the Improvement of Geometrical Teaching; Member of the Edinburgh Mathematical Society; **Honorary Secretary**; 27 *Cantlowes Road, Camden Square, N.W.*

1877 Jan. 11. Tunzelmann, George William von, B.Sc. London; Member of the Physical Society; *Devon Lodge, Dartmouth, South Devon.*

1883 Dec. 13. Turner, Herbert Hall, B.A. Cambridge, B.Sc. London; Fellow of Trinity College, Cambridge; *Royal Observatory, Greenwich.*

1865 Oct. 16. Twisden, Rev. John Francis, M.A. Cambridge; Professor of Mathematics in the Royal Military Staff College; *Roydon Lodge, Camberley, Farnborough, Hants.*

†1881 May 12. Van Velzer, Charles Ambrose, Ph.D.; Professor of Mathematics in the University of Wisconsin; *Madison, Wisconsin, U.S.A.*

Date of Election.

1866 Jan. 15. **Vinter**, Rev. **Frederick William**, **M.A.**; late Fellow of St. John's College, Cambridge; *Glenville, Walton-on-Thames, Surrey.*

†1865 Oct. 16. WALKER, JOHN JAMES, M.A. Dublin; F.R.S.; Member of the Physical Society; VICE-PRESIDENT; *Cannycot, Willoughby Road, Hampstead, N.W.*

1869 April 8. Walmsley, John, B.A. London; *The Briers, Winton, Patricroft, Manchester.*

1883 Nov. 8. Watkin, Frederick William, B.A.; late Scholar of Corpus Christi College, Oxford; *St. Paul's School, Hammersmith, W.*

1873 Dec. 11. Webb, Robert Rumsey, M.A.; Fellow of St. John's College, and Lecturer at St. John's and Emmanuel Colleges, Cambridge; *Cambridge.*

1869 Dec. 9. White, Rev. James, M.A. Trinity College, Dublin; Head-Master and Chaplain of the Royal Naval School, *New Cross.*

1873 Dec. 11. Wilkinson, Rev. Michael Marlow Umfreville, M.A., F.C.P.S.; late Fellow and Assistant Tutor of Trinity College, Cambridge; *The Rectory, Reepham-cum-Kerdiston, Norwich.*

1875 Jan. 14. Williamson, Benjamin, M.A., F.R.S.; Fellow and Tutor of Trinity College, Dublin; Professor of Natural Philosophy in the University of Dublin; *Dublin.*

1871 Feb. 9. Wolstenholme, Rev. Joseph, M.A., D.Sc., Cambridge; F.C.P.S.; late Fellow of Christ's College, Cambridge; Professor of Mathematics in the Royal Indian Engineering College; *Coopers Hill, Staines.*

1882 June 8. Wood, James, M.A.; Fellow of Queen's College, Cambridge; Head Mathematical Master of the Glasgow Academy; *Glasgow.*

1882 Dec. 14. Woodcock, Thomas, B.A.; late Scholar of Christ's College, Cambridge; *Holly House, Twickenham.*

1867 Dec. 12. Woolhouse, Westley Stoker Barker, F.R.A.S., F.I.A., F.S.S.; *Alwyne Lodge, Canonbury, N.*

†1865 Oct. 16. Wren, Walter, M.A. Christ's College, Cambridge; 3 *Powis Square, Westbourne Park, W.*

†1876 Mar. 16. Wright, Richard Thomas, M.A.; late Fellow and Tutor of Christ's College, Cambridge; 3 *Dr. Johnson's Buildings, Temple, E.C.*

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